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Amplitudes on Entities

Stanley Gudder[1]

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A stochastic model for a quantum system is developed in terms of amplitude densities on an entity. An entity provides an axiomatic description for the set of tests and states of a physical system and an amplitude density gives a means for computing probabilities within this framework. The interference and independence of tests relative to an amplitude density are formulated. Various ways of combining entities and amplitudes are presented. Superpositions of amplitudes and superselection sectors in the amplitude space are considered. Finally, symmetry groups and systems of covariance on an entity are developed.

1. INTRODUCTION

Over the past 20 years, Foulis and Randall (1972*a*,*b*, 1983; Foulis *et al.*, 1983; Foulis, 1989; Bennett and Foulis, to appear) have formulated a framework for operational statistics. Their intention was to develop a language capable of discussing and comparing theories for the empirical sciences. The latest and most elegant formulation for operational statistics is based on the concept of an entity (Foulis *et al.*, 1983; Foulis, 1989; Foulis and Bennett, to appear). An entity provides an axiomatic description of the tests and states for a physical system. The tests correspond to physical observables, experiments, or measurements, while the states correspond to the condition or preparation of the physical system. Although we are free to perform any tests that are within our capabilities, the states are restricted to those that are allowed by nature.

An entity alone does not provide a complete description of a stochastic model for a physical system. We are still missing a method for computing probabilities in the model. In the present paper, this is accomplished by introducing amplitudes on an entity. Following ideas of Feynman and Dirac

[1]Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208.

(Feynman, 1948; Feynman and Hibbs, 1965), we define an amplitude density as a complex-valued function with certain properties defined on the set of states. The amplitude of an outcome of a test is then defined as the sum of the amplitudes of the states that guarantee that outcome. The probability that an outcome occurs, when tested, is the absolute value squared of its amplitude. The combination of an amplitude density with an entity then provides a stochastic model which appears to give a satisfactory quantum mechanical description of a physical system.

It is sometimes argued that complex amplitudes have no direct physical significance and hence they should not be included in a basic physical theory. This may be true in the sense that complex amplitudes are not seen in the laboratory. However, probabilities do have physical significance and they are the main contact between theory and experiment. The important question is, How are these probabilities computed? It appears that in quantum mechanics, they must be computed by summing amplitudes and then taking the absolute value squared. If this is the case, amplitudes can have a place as a primitive concept in a quantum theory. As an analogy, no one has observed a colored particle, yet it is generally accepted that quarks (which are themselves only indirectly observed) must have a property called color. This property is responsible for an internal symmetry group which is basic to quantum chromodynamics. Although color is never observed in the laboratory, physicists do not hesitate to incorporate it in a basic theory of elementary particles.

In Section 2, we set the notation and give the basic definitions that will be used in the sequel. Various examples are given to illustrate the concepts that are introduced. Section 3 considers interference and independence of tests relative to an amplitude density. It is shown that these two concepts are unrelated. Distributions for tests and probabilities of events are defined. Hilbertian entities are studied and a comparison is made between the present framework and that of conventional Hilbert space quantum mechanics. Section 4 presents methods for combining entities to form new entities. In particular, the horizontal sum, the direct sum, and the Cartesian product of entities are discussed. Moreover, relationships between amplitudes on a combined entity and those on the component entities are derived. In Section 5, we introduce the concept of a sector in the amplitude space of an entity. Sectors are related to the superposition principle and they describe amplitudes for which a superposition is possible. Considered as a collection of sectors, the amplitude space becomes a partial Hilbert space. The sector structure of a simple, but nontrivial, example is derived. Sectors for direct sums and Cartesian products are considered. Section 6 presents symmetry groups and systems of covariance on an entity. It is shown that a test can be represented by a positive operator-valued (POV) measure from its set of events into a sector. Tests that can be represented by a projection-valued measure are characterized. Generalized unitary representations of a symmetry group are introduced and it is shown that the POV measure corresponding to a test provides a system of covariance for such a representation.

For alternative approaches to quantum probability, we refer the reader to Gudder (1988*a*) and the references cited therein. Besides its great generality, one of the advantages of the present approach is that it provides a large number of finite examples that can improve our intuition concerning quantum probability.

2. NOTATION AND DEFINITIONS

Let $\mathscr{S}$ be a physical system and let $\hat{E}$ be an experiment that can be performed on $\mathscr{S}$. Each time $\hat{E}$ is performed, an outcome is obtained and we denote this set of possible outcomes by E. We call E a *test* and when $\hat{E}$ is performed, we say that the test E has been executed. In general, various experiments can be performed on $\mathscr{S}$ and consequently there is an associated collection of tests $\mathscr{A}$. We call the union $X=\bigcup \mathscr{A}$ of these tests an *outcome set*. A *test space* is a pair $(X, \mathscr{A})$ where $\mathscr{A}$ is a nonempty collection of nonempty sets satisfying

$$X=\bigcup \mathscr{A} \tag{2.1}$$

$$\text{If} \quad E, F\in\mathscr{A} \quad \text{with} \quad E\subseteq F, \quad \text{then} \quad E=F \tag{2.2}$$

Condition (2.2) is called *irredundancy*. This mild condition is imposed since there is no need to include a test that is properly contained in another test. A subset A of a test $E\in\mathscr{A}$ is called an *event*. We denote the set of all events in E by $\mathscr{E}(E)$ and the set of all events by $\mathscr{E}(\mathscr{A})$. Thus, $\mathscr{E}(E)=2^E$ and

$$\mathscr{E}(\mathscr{A})=\bigcup_{E\in\mathscr{A}} 2^E$$

An event is *proper* if it is nonempty and not equal to a test.

In the sequel, $(X, \mathscr{A})$ will denote a test space for a physical system $\mathscr{S}$. Suppose $\mathscr{S}$ is prepared to be in a certain condition $\hat{S}$ and let $S\subseteq X$ be the set of outcomes that are possible under this condition. If $A\in\mathscr{E}(E)$ and $S\cap E\subseteq A$, then the event A must occur when E is executed and $\mathscr{S}$ is in condition $\hat{S}$. Now suppose we also have $A\in\mathscr{E}(F)$ for some other test $F\in\mathscr{A}$. If F is executed, then for consistency, A should again occur; that is, the occurrence of an event should be independent of the executed test containing that event. We call $S\subseteq X$ a *support* if

$$A\in\mathscr{E}(E)\cap\mathscr{E}(F) \quad \text{and} \quad S\cap E\subseteq A \quad \text{imply} \quad S\cap F\subseteq A \tag{2.3}$$

We denote the set of all supports in $(X, \mathscr{A})$ by $\hat{\Sigma}(\mathscr{A})$. Notice that $\varnothing$, $X \in \hat{\Sigma}(\mathscr{A})$ for any test space $(X, \mathscr{A})$ We denote the set of all nonempty supports by $\Sigma(\mathscr{A})$. If $A \in \mathscr{E}(E)$, $S \in \Sigma(\mathscr{A})$, and $S \cap E \subseteq A$ we say that S *guarantees* A. A set $S \subseteq X$ satisfies the *exchange condition* if

$$E, F \in \mathscr{A} \quad \text{and} \quad S \cap E \subseteq F \quad \text{imply} \quad S \cap F \subseteq E \tag{2.4}$$

The next lemma shows that (2.3) and (2.4) are equivalent.

Lemma 2.1. (a) A subset $S \subseteq X$ is a support if and only if S satisfies the exchange condition. (b) If $S \in \Sigma(\mathscr{A})$, then $S \cap E \neq \varnothing$ for all $E \in \mathscr{A}$.

Proof. (a) Suppose S satisfies (2.4), $A \in \mathscr{E}(E) \cap \mathscr{E}(F)$, and $S \cap E \subseteq A$. Then $S \cap E \subseteq F$, so by (2.4), $S \cap F \subseteq E$. Since $S \cap F \subseteq S$, we have

$$S \cap F \subseteq S \cap E \subseteq A$$

Hence, (2.3) holds and $S \in \hat{\Sigma}(\mathscr{A})$. Conversely, suppose $S \in \hat{\Sigma}(\mathscr{A})$, E, $F \in \mathscr{A}$, and $S \cap E \subseteq F$. Letting $A = S \cap E$, we have $A \subseteq F$, so $A \in \mathscr{E}(E) \cap \mathscr{E}(F)$. Applying (2.3) gives

$$S \cap F \subseteq A \subseteq E$$

Hence, (2.4) holds. (b) Suppose $S \in \Sigma(\mathscr{A})$ and $S \cap E = \varnothing$ for some $E \in \mathscr{A}$. Since $S \neq \varnothing$ and $X = \bigcup \mathscr{A}$, there exists an $F \in \mathscr{A}$ such that $S \cap F \neq \varnothing$. Then $S \cap E \subseteq F$, so by (2.4), $S \cap F \subseteq E$. Hence, $S \cap F \subseteq S \cap E$, so $S \cap E \neq \varnothing$. This gives a contradiction. ■

We can interpret Lemma 2.1(b) as follows. If $\mathscr{S}$ is in the condition $\hat{S}$ and some outcome is possible, then when any test is executed, an outcome must occur. It is straightforward to show that $\hat{\Sigma}(\mathscr{A})$ is a complete lattice under set-theoretic inclusion and that the supremum of a collection of supports in this lattice coincides with their union. We shall later discuss a sublattice of $\hat{\Sigma}(\mathscr{A})$ called the property (or attribute) lattice. A property will then be interpreted as a special kind of support that specifies a physical property of the system.

A *probability weight* on $(X, \mathscr{A})$ is a function $\mu : X \to [0, 1] \subseteq \mathbb{R}$ such that $\sum_{x \in E} \mu(x) = 1$ for all $E \in \mathscr{A}$. We interpret $\mu(x)$ as the probability that the outcome x occurs when a test containing x is executed. If $A \in \mathscr{E}(\mathscr{A})$, we define $\mu(A) = \sum_{x \in A} \mu(x)$ and interpret $\mu(A)$ as the probability that the event A occurs when tested. We denote the set of probability weights on $(X, \mathscr{A})$ by $\Omega(\mathscr{A})$. For $\mu \in \Omega(\mathscr{A})$, we define

$$\operatorname{supp} \mu = \{x \in X : \mu(x) > 0\}$$

Of course, $\operatorname{supp} \mu \neq \varnothing$ for any $\mu \in \Omega(\mathscr{A})$.

Lemma 2.2. If $S = \operatorname{supp} \mu$ for some $\mu \in \Omega(\mathscr{A})$, then $S \in \Sigma(\mathscr{A})$.

Proof. Suppose $E, F \in \mathscr{A}$ and $S \cap E \subseteq F$. Assume $S \cap F \nsubseteq E$. Then there exists $y \in S \cap F$ with $y \notin E$. Now $y \notin S \cap E$ and

$$\{y\} \cup (S \cap E) \subseteq S \cap F$$

Hence,

$$1 = \sum_{x \in S \cap F} \mu(x) \geq \sum_{x \in S \cap E} \mu(x) + \mu(y) = 1 + \mu(y)$$

This implies that $\mu(y) = 0$, which is a contradiction. ■

We call $S \in \Sigma(\mathscr{A})$ a *stochastic support* if $S = \operatorname{supp} \mu$ for some $\mu \in \Omega(\mathscr{A})$. Most of the supports that we shall consider in examples are stochastic supports. In such cases, the easiest way to show that $S \in \Sigma(\mathscr{A})$ is usually obtained by showing that $S = \operatorname{supp} \mu$ for some $\mu \in \Omega(\mathscr{A})$. The next two examples show that there exist nonempty nonstochastic supports.

Example 2.1 (Bowtie). Let $(X, \mathscr{A})$ be the test space with $X = \{x_1, \ldots, x_5\}$ and $\mathscr{A} = \{E, F, G, H\}$, where $E = \{x_2, x_3, x_4\}$, $F = \{x_1, x_3, x_5\}$, $G = \{x_1, x_2\}$, $H = \{x_4, x_5\}$. It is easy to check that the only nonempty supports are $S = \{x_1, x_2, x_4, x_5\}$, $T = \{x_1, x_4\}$, $U = \{x_2, x_5\}$, $V = X$. Moreover, every $\mu \in \Omega(\mathscr{A})$ has the form $\mu(x_1) = \mu(x_4) = a$, $\mu(x_2) = \mu(x_5) = 1 - a$, $\mu(x_3) = 0$ for some $a \in [0, 1]$. It follows that S, T, U are stochastic supports and V is a nonstochastic support. ■

Example 2.2 (D. Foulis). Let $\mathbb{R}^3$ be the standard 3-dimensional Euclidnean space and let X be the unit sphere in $\mathbb{R}^3$. Letting $\mathscr{A}$ be the set of all orthonormal bases in $\mathbb{R}^3$, we see that $(X, \mathscr{A})$ is a test space. Applying Gleason's theorem (Gleason, 1957), we find that every $\mu \in \Omega(\mathscr{A})$ has the form $\mu(x) = \langle Tx, x \rangle$, where T is a positive operator of trace 1. It follows that every stochastic support is the set-theoretic complement of a subspace intersected with X. Let $y, z \in X$ be linearly independent but not orthogonal and let S be the set-theoretic complement of $\{y, -y, z, -z\}$ in X. Then S is not a stochastic support. However, we now show that S is a support. Suppose $E, F \in \mathscr{A}$ and $S \cap E \subseteq F$. If $S \cap E = E$, then $E \subseteq F$. Hence, $E = F$ and $S \cap F \subseteq E$. If $S \cap E \neq E$, then exactly one of the vectors y, $-y$, z, $-z$ is in E. Suppose $y \in E$ and $E = \{y, x_1, x_2\}$. Then $S \cap E = \{x_1, x_2\} \subseteq F$, so $F = \{x, x_1, x_2\}$ for some $x \in X$. It follows that $x = \pm y$, so $S \cap F = \{x_1, x_2\} \subseteq E$. The other cases are similar. ■

In practice, there may be supports that do not correspond to a physically realizable condition or we may want to distinguish a convenient set of supports. For these reasons, we frequently consider only a sufficiently rich subset of $\Sigma(\mathscr{A})$. An *entity* is a triple $(X, \mathscr{A}, \Sigma)$, where $(X, \mathscr{A})$ is a test space and Σ is a collection of nonempty supports that covers X (that is, $X = \bigcup \Sigma$).

We call the elements of Σ *states*. The previous conditions imply that at least one outcome is possible for a given state and every outcome is possible for some state. We frequently denote an entity $(X, \mathscr{A}, \Sigma)$ simply as X. The following lemma gives a concise characterization of an entity (Bennett and Foulis, to appear).

Lemma 2.3. Let X be a nonempty set and let $\mathscr{A}$, Σ be two collections of nonempty subsets of X. Then $(X, \mathscr{A}, \Sigma)$ is an entity if and only if the following two conditions hold. (a) For every $x \in X$, there exist $E \in \mathscr{A}$, $S \in \Sigma$ such that $x \in E \cap S$. (b) If E, $F \in \mathscr{A}$ and $S \in \Sigma$, then $S \cap E \subseteq F$ implies $S \cap F \subseteq E$.

Proof. If $(X, \mathscr{A}, \Sigma)$ is an entity, then $X = \bigcup \mathscr{A} = \bigcup \Sigma$ implies (a) and (2.4) together with Lemma 2.1(a) imply (b). Conversely, suppose $(X, \mathscr{A}, \Sigma)$ satisfies (a) and (b). The only condition for an entity requiring proof is (2.2). Suppose E, $F \in \mathscr{A}$ with $E \subseteq F$ and let $x \in F$. By (a), there exists an $S \in \Sigma$ such that $x \in S$. Since $S \cap E \subseteq F$, applying (b) gives $x \in S \cap F \subseteq E$. Hence, $F \subseteq E$, so $E = F$. ■

A *property* of an entity X is a union of states in X. Although a property may not be a state, it is always a support. We denote the complete lattice of properties of X by $\mathscr{L}(X)$. The *property lattice* $\mathscr{L}(X)$ is important for investigations of the quantum logic structure of X (Foulis and Randall, 1983; Foulis *et al.*, 1983; Foulis, 1989). If $A \in \mathscr{E}(\mathscr{A})$, we denote by $[A]$ the union of the states that guarantee A. Thus, $[A] \in \mathscr{L}(X)$ and $[A]$ is the largest property that guarantees A. For $x \in X$, we write $[x] = [\{x\}]$. We call X *unital* if for every $x \in X$ there exists an $S \in \Sigma$ such that $S \subseteq [x]$; that is, S guarantees x. It follows that if X is unital and $A \in \mathscr{E}(\mathscr{A})$ with $A \neq \emptyset$, then there is an $S \in \Sigma$ such that $S \subseteq [A]$. In Example 2.1, $(X, \mathscr{A}, \Sigma(\mathscr{A}))$ is an entity that is not unital, since no state guarantees x_3. In Example 2.2, $(X, \mathscr{A}, \Sigma(\mathscr{A}))$ is a unital entity. The simplest example of an entity is the *singular entity* $(X_0, \mathscr{A}_0, \Sigma_0)$, where $X_0 = \{x\}$, $\mathscr{A}_0 = \Sigma_0 = \{\{x\}\}$. Of course, this entity is unital.

In the sequel, $X = (X, \mathscr{A}, \Sigma)$ will denote an entity. In order to compute probabilities of events for X we must endow X with a quantum probability structure. This is accomplished by introducing an amplitude function $f: \Sigma \to \mathbb{C}$. As in traditional probability theory, the amplitude function f provides a stochastic model for our physical system. Following ideas of Feynman (1948; Feynman and Hibbs, 1965), we interpret $f(S)$, $S \in \Sigma$, as the amplitude that the system is in state S. Moreover, the amplitude of an outcome x is the sum of the amplitudes of the states that result in x with certainty when x is tested. Finally, the probability of x is the absolute value squared of its amplitude. The author has used these same ideas in previous developments of quantum probability theory (Gudder, 1988*b*,*c*, 1989). The reader should note that a probability weight μ does not give an adequate stochastic model, since μ is independent of the state structure. For this reason, μ does not adequately describe quantum interference phenomena. As we shall see, f does induce a probability weight, although the converse does not hold in general. With this motivation, we now give precise definitions for these concepts.

A function $f\colon \Sigma \to \mathbb{C}$ is *summable* if for every $x \in X$.

$$\sum_{S \subseteq [x]} |f(s)| < \infty \tag{2.5}$$

If f is summable and $x \in X$, we define

$$\hat{f}(x) = \sum_{S \subseteq [x]} f(S) \tag{2.6}$$

Notice that if $x \in E \in \mathscr{A}$, then

$$\hat{f}(x) = \sum \{f(S)\colon S \cap E = x\} \tag{2.7}$$

and it follows from (2.3) that (2.7) does not depend on the test E containing x. A summable function $f\colon \Sigma \to \mathbb{C}$ is an *amplitude* if for every $E, F \in \mathscr{A}$ we have

$$\sum_{x \in E} |\hat{f}(x)|^2 = \sum_{x \in F} |\hat{f}(x)|^2 < \infty \tag{2.8}$$

If f is an amplitude, we define $\|f\|^2 = \sum_{x \in E} |\hat{f}(x)|^2$ and of course $\|f\|$ is independent of $E \in \mathscr{A}$. We denote the set of all amplitudes on X by $\mathscr{H}(X)$ and call $\mathscr{H}(X)$ the *amplitude space* for X. An amplitude f is an *amplitude density* if $\|f\| = 1$ and we denote the set of all amplitude densities on X by $\mathscr{D}(X)$. Of course, if $f \in \mathscr{H}(X)$ with $\|f\| \neq 0$ then $f/\|f\| \in \mathscr{D}(X)$. Although $\mathscr{D}(X)$ is important for computing probabilities, it is sometimes more convenient to consider $\mathscr{H}(X)$ because of its linear structure. Notice that if $f \in \mathscr{D}(X)$, then $\mu(x) = |\hat{f}(x)|^2$ is a probability weight on X. We interpret $|\hat{f}(x)|^2$ as the probability that x occurs when tested in the stochastic model provided by f. We say that $\mu \in \Omega(\mathscr{A})$ is *induced* if there exists an $f \in \mathscr{D}(X)$ such that $\mu(x) = |\hat{f}(x)|^2$ for every $x \in X$. For the entity $(X, \mathscr{A}, \Sigma(\mathscr{A}))$ in Example 2.1, every $\mu \in (\mathscr{A})$ is induced. Indeed, for $a \in [0, 1]$, define $f(T) = a^{1/2}$, $f(U) = (1-a)^{1/2}$, $f(S) = f(V) = 0$. Then $\hat{f}(x_1) = \hat{f}(x_4) = a^{1/2}$, $\hat{f}(x_2) = \hat{f}(x_5) = (1-a)^{1/2}$, $\hat{f}(x_3) = 0$, so the general probability weight is induced by f. If every $\mu \in \Omega(\mathscr{A})$ is induced, we say that $\Omega(\mathscr{A})$ is *induced.*

Example 2.3 (Little Triangle). Let $(X, \mathscr{A})$ be the test space with $X = \{x_1, x_2, x_3\}$ and $\mathscr{A} = \{E, F, G\}$, where $E = \{x_1, x_2\}$, $F = \{x_2, x_3\}$, and $G = \{x_1, x_3\}$. The only nonempty support in $(X, \mathscr{A})$ is $S = X$, so this test space generates a unique entity $(X, \mathscr{A}, \Sigma)$, where $\Sigma = \Sigma(\mathscr{A}) = \{S\}$. There is only one $\mu \in \Omega(\mathscr{A})$ and this is given by $\mu(x) = 1/2$ for all $x \in X$. Since $[x] = \varnothing$ for all $x \in X$, it is clear that $\mathscr{D}(X) = \varnothing$. Hence, μ is not induced. ■

Example 2.4 (Wedge). Let $(X, \mathscr{A}, \Sigma(\mathscr{A}))$ be the entity with $X = \{x_1, \ldots, x_7\}$ and $\mathscr{A} = E, F, G, H$, where $E = \{x_1, x_2, x_3\}$, $F = \{x_1, x_5, x_7\}$, $G = \{x_2, x_5, x_6\}$, and $H = \{x_3, x_4, x_5\}$. Let $\mu \in \Omega(\mathscr{A})$ be defined by $\mu(x) = 1/3$ for all $x \in X$. It is easy to check that $[x_5] = \varnothing$. Hence, $\hat{f}(x_5) = 0$ for all $f \in \mathscr{D}(X)$. It follows that μ is not induced. ■

Example 2.5 (Wright Triangle). The previous two examples were not unital. We now consider the unital entity $(X, \mathscr{A}, \Sigma)$ with $X = \{x_1, \ldots, x_6\}$,

$$\mathscr{A} = \{E, F, G\}$$

(where $E = \{x_1, x_2, x_3\}$, $F = \{x_3, x_4, x_5\}$, and $G = \{x_5, x_6, x_1\}$), and

$$\Sigma = \{S, T, U\}$$

(where $S = \{x_1, x_4\}$, $T = \{x_2, x_5\}$, $U = \{x_3, x_6\}$). Let $\mu \in \Omega(\mathscr{A})$ be defined by $\mu(x_2) = \mu(x_4) = \mu(x_6) = 1$, $\mu(x_1) = \mu(x_3) = \mu(x_5) = 0$. Suppose μ is induced by $f \in \mathscr{D}(X)$. Then

$$1 = \mu(x_2) = |\hat{f}(x_2)|^2 = |f(T)|^2$$

and

$$0 = \mu(x_5) = |\hat{f}(x_5)|^2 = |f(T)|^2$$

This is a contradiction, so μ is not induced. ■

3. INTERFERENCE AND INDEPENDENCE

For $E \in \mathscr{A}$ we define the *E-Hilbert space*

$$\mathscr{H}_E = l^2(E) = \left\{ g: E \to \mathbb{C}: \sum_{x \in E} |g(x)|^2 < \infty \right\}$$

Of course, addition and scalar multiplication in $\mathscr{H}_E$ are defined pointwise and the inner product is given by

$$\langle g, h \rangle_E = \sum_{x \in E} g(x) \bar{h}(x)$$

If $f \in \mathscr{H}(X)$, $E \in \mathscr{A}$, we define the *(E, f)-wave function* f_E by $f_E = \hat{f} \mid E$. Thus, $f_E: E \to \mathbb{C}$ with $f_E(x) = \hat{f}(x)$ for all $x \in E$. Notice that $f_E \in \mathscr{H}_E$ and the l^2-norm

$\|f_E\|_E$ equals $\|f\|$. Moreover, if $f\in\mathscr{D}(X)$, then $\|f_E\|_E=1$. For $A\in\mathscr{E}(\mathscr{A})$, $x\in X$, $f\in\mathscr{H}(X)$, we define

$$\hat{f}(A)(x)=\sum\{f(S): S\subseteq[x]\wedge[A]\} \tag{3.1}$$

It follows from (2.5) that $\hat{f}(A)(x)$ exists. If $x\in E$, $A\subseteq F$, E, $F\in\mathscr{A}$, we can rewrite (3.1) as

$$\hat{f}(A)(x)=\sum\{f(S): S\cap E=x, S\cap F\subseteq A\} \tag{3.2}$$

and the expression in (3.2) is independent of the tests containing x and A. Notice that for $F\in\mathscr{A}$, we have $\hat{f}(F)(x)=\hat{f}(x)$. If $f\in\mathscr{D}(X)$, we interpret $\hat{f}(A)(x)$ as the amplitude that A and x both occur. For $E\in\mathscr{A}$, $f\in\mathscr{D}(X)$, $A\in\mathscr{E}(\mathscr{A})$, we define the (E,f)-*pseudoprobability* of A by

$$P_{E,f}(A)=\sum_{x\in E}|\hat{f}(A)(x)|^2 \tag{3.3}$$

It is clear that $P_{E,f}(A)\geq 0$ and $P_{E,f}(F)=1$ for all $F\in\mathscr{A}$.

In general, $P_{E,f}(A)$ cannot be interpreted as a probability, since it can be larger than 1 and need not be additive on $\mathscr{E}(F)$, $F\in\mathscr{A}$. If $x\in E$, $A\subseteq E$, then applying (3.2) with $F=E$ gives

$$\hat{f}(A)(x)=\chi_A\hat{f}(x)$$

Hence, in this case, we have

$$P_{E,f}(A)=\sum_{x\in E}\chi_A(x)|\hat{f}(x)|^2=\sum_{x\in A}|\hat{f}(x)|^2=\sum_{x\in A}|f_E(x)|^2$$

We conclude that $P_{E,f}$ is a probability measure on $\mathscr{A}(E)$ and we call $P_{E,f}|\mathscr{E}(E)$ the *f-distribution* of E. We say that E *does not interfere with F relative to f* if $P_{E,f}(A)=P_{F,f}(A)$ for every $A\in\mathscr{E}(F)$. We interpret this as saying that if E is used to test F events, then the same distribution is obtained as when F itself is employed. In this case E contains complete statistical information concerning F in the model provided by f. Moreover, $P_{E,f}$ gives a probability measure on $\mathscr{E}(F)$ as well as on $\mathscr{E}(E)$. Example 4.1 will show that noninterference is not a symmetric relation in general.

In traditional probability theory there is never interference between tests. For simplicity, let $(\Omega,\mathscr{F},\mu)$ be a finite probability space and let $E:\Omega\to\mathbb{R}$ be a random variable. If $A\in\mathscr{F}$ and x is in the range of E, then the probability that A and x occur is

$$P(A,x)=\mu[A\cap E^{-1}(x)]$$

In analogy with (3.3), we would have

$$P_{E,\mu}(A)=\sum_x\mu[A\cap E^{-1}(x)]=\mu(A)$$

Hence, if F is another random variable, we would have $P_{E,\mu}(A)=P_{F,\mu}(A)$ for all $A\in\mathscr{F}$.

There is another important case in which we have no interference. A support S is called *dispersion-free* if $|S\cap E|=1$ for every $E\in\mathscr{A}$. We denote the set of dispersion-free supports by Σ_d. For example, T, and U in Example 2.1 are dispersion-free and S, T, U in Example 2.5 are dispersion-free. We call $f\in\mathscr{D}(X)$ *dispersion-free* if there exists an $S\in\Sigma_d\cap\Sigma$ such that $f(S)=1$, $f(T)=0$ for $T\in\Sigma$, $T\neq S$. Then for $x\in X$, $A\in\mathscr{E}(\mathscr{A})$ we have

$$\hat{f}(A)(x)=\begin{cases}1 & \text{if} \quad x\in S, \quad S\cap A\neq\varnothing\\ 0 & \text{otherwise}\end{cases}$$

If $E\in\mathscr{A}$, we obtain

$$P_{E,f}(A)=\sum_{x\in E}|\hat{f}(A)(x)|^2=|\hat{f}(A)(y)|^2$$

where $y=S\cap E$. Hence,

$$P_{E,f}(A)=\begin{cases}1 & \text{if} \quad S\cap A\neq\varnothing\\ 0 & \text{otherwise}\end{cases}$$

It follows that any two tests do not interfere relative to a dispersion-free density.

Let $B\subseteq E\in\mathscr{A}$, $f\in\mathscr{D}(X)$, and suppose $A\in\mathscr{E}(\mathscr{A})$ with $0<P_{E,f}(A)<\infty$. We then define the *conditional probability*

$$P_{E,f}(B|A)=\frac{\sum_{x\in B}|\hat{f}(A)(x)|^2}{P_{E,f}(A)} \tag{3.4}$$

Notice that $P_{E,f}(\cdot|A)$ is a probability measure on $\mathscr{E}(E)$. If $A\subseteq E$, then (3.4) reduces to

$$P_{E,f}(B|A)=\frac{P_{E,f}(B\cap A)}{P_{E,f}(A)} \tag{3.5}$$

and (3.5) is the usual form for a conditional probability. Moreover, if $A=F\in\mathscr{A}$, then (3.4) becomes

$$P_{E,f}(B|F)=P_{E,f}(B)$$

We say that E is *independent of* F relative to f if $P_{E,f}(B|A)=P_{E,f}(B)$ whenever $0<P_{E,f}(A)<\infty$ for every $B\subseteq E$, $A\subseteq F$. It follows that E is independent of F relative to f if and only if for every $B \subseteq E$, $A \subseteq F$ with $0 < P_{E,f}(A) < \infty$ we have

$$\sum_{x \in B} |\hat{f}(A)(x)|^2 = P_{E,f}(B) P_{E,f}(A) \tag{3.6}$$

For example, let $f \in \mathscr{D}(X)$ be dispersion-free with $f(S) = 1$. Then both sides of (3.6) equal 1 if $S \cap B \neq \varnothing$ and $S \cap A \neq \varnothing$ and they equal 0 otherwise. Hence, any two tests are independent relative to f. Notice that in general, both sides of (3.6) agree if A or B is not proper.

Example 3.1 (Firefly). Let $(X, \mathscr{A}, \Sigma)$ be the entity with $X = \{x_1, x_2, x_3, x_4\}$, $\mathscr{A} = \{E, F\}$ (where $E = \{x_1, x_2\}$ and $F = \{x_3, x_4\}$, and $\Sigma = \{S, T, U, V\}$ (where $S = \{x_1, x_4\}$, $T = \{x_2, x_4\}$, $U = \{x_1, x_3\}$, and $V = \{x_2, x_3\}$). If $f \in \mathscr{D}(X)$ and $f(S) = a$, $f(T) = b$, $f(U) = c$, $f(V) = d$, we have $\hat{f}(x_1) = a + c$, $\hat{f}(x_2) = b + d$, $\hat{f}(x_3) = c + d$, $\hat{f}(x_4) = a + b$. Of course, since $f \in \mathscr{D}(X)$, we have

$$|a+c|^2 + |b+d|^2 = |c+d|^2 + |a+b|^2 = 1$$

Moreover, $\hat{f}(\{x_3\})(x_1) = c$, $\hat{f}(\{x_3\})(x_2) = d$, $\hat{f}(\{x_4\})(x_1) = a$, $\hat{f}(\{x_4\})(x_2) = b$. Hence,

$$P_{E,f}(\{x_3\}) = |c|^2 + |d|^2$$
$$P_{E,f}(\{x_4\}) = |a|^2 + |b|^2$$

It follows that E does not interfere with F relative to f if and only if

$$|c|^2 + |d|^2 = |c+d|^2, \qquad |a|^2 + |b|^2 = |a+b|^2.$$

Moreover, E is independent of F relative to f if and only if

$$|c|^2 = |a+c|^2[|c|^2 + |d|^2]$$
$$|a|^2 = |a+c|^2[|a|^2 + |b|^2]$$
$$|d|^2 = |b+d|^2[|c|^2 + |d|^2]$$
$$|b|^2 = |b+d|^2[|a|^2 + |b|^2]$$

Let $f_1 : \Sigma \to \mathbb{C}$ be defined by $a = 1$, $b = c = d = 0$. Since S is dispersion-free, so is f_1. It follows that E and F do not interfere and are independent relative to f_1. Let $f_2 : \Sigma \to \mathbb{C}$ be defined by $a = b = c = -d = 1/2$. It is easy to check that $f_2 \in \mathscr{D}(X)$ and E interferes with F relative to f_2. Moreover,

$$|c|^2 = \tfrac{1}{4} \neq \tfrac{1}{2} = |a+c|^2[|c|^2 + |d|^2]$$

so E is not independent of F relative to f_2. Finally, let $f_3: \Sigma \to \mathbb{C}$ be defined by $a=d=i/\sqrt{10}$, $b=c=2/\sqrt{10}$. It is easy to check that $f_3 \in \mathscr{D}(X)$ and E does not interfere with F relative to f_3. Moreover,

$$|c|^2=\tfrac{2}{5}\neq\tfrac{1}{4}=|a+c|^2[|c|^2+|d|^2]$$

so E is not independent of F relative to f_3. ■

Example 3.2 (Hilbertian Entity). This example corresponds to the traditional Hilbert space formulation of quantum mechanics. Let $\mathscr{H}$ be a complex Hilbert space. From each 1-dimensional subspace of $\mathscr{H}$ choose precisely one unit vector and let X be the set of all these unit vectors. Let $\mathscr{A}$ be the collection of all maximal orthogonal sets in X and form the test space $(X, \mathscr{A})$. For $x \in X$, let $S_x=\{y \in X: \langle x, y\rangle \neq 0\}$ and let $\Sigma=\{S_x: x \in X\}$. For $x \in X$, define $\mu_x: X \to [0, 1]$ by $\mu_x(y)=|\langle x, y\rangle|^2$. Then $\mu_x \in \Omega(\mathscr{A})$ and $S_x = \operatorname{supp} \mu_x$. Applying Lemma 2.2, we have that $S_x \in \Sigma(\mathscr{A})$ for all $x \in X$. It follows that $(X, \mathscr{A}, \Sigma)$ is a unital entity. For $x \in X$, define $f_x: \Sigma \to \mathbb{C}$ by $f_x(S_y)=\langle x, y\rangle$. This is well-defined, since $S_y \neq S_z$ if $y \neq z$; that is, $y \mapsto S_y$ is a bijection from X to Σ. Notice that $S_x \subseteq [y]$ if and only if $x=y$. Hence,

$$\hat{f}_x(y)=\sum \{f_x(S_z): S_z \subseteq [y]\}=f_x(S_y)=\langle x, y\rangle \tag{3.7}$$

and for $E \in \mathscr{A}$ we have

$$\sum_{y \in E} |\hat{f}_x(y)|^2=\sum_{y \in E} |\langle x, y\rangle|^2=1$$

It follows that $f_x \in \mathscr{D}(X)$ for all $x \in X$ and $f_x(S_x)=1$. Moreover, if $\dim \mathscr{H} \geq 3$, then f_x is essentially the only amplitude density with this property. Indeed, suppose $f \in \mathscr{D}(X)$ and $f(S_x)=1$. Then, as in (3.7), $f(S_y)=\hat{f}(y)$. It follows from the proof of Gleason's theorem (Gleason, 1957) that there exists a unique positive trace class operator T of trace 1 on $\mathscr{H}$ such that $|\hat{f}(y)|^2= \langle Ty, y\rangle$ for all $y \in X$. Moreover, since $\hat{f}(x)=1$, we conclude that T is the 1-dimensional projection onto the span of x. Hence, $|\hat{f}(y)|^2=|\langle x, y\rangle|^2$, so there exists a function $\phi: X \to \mathbb{R}$ such that

$$f(S_y)=\hat{f}(y)=e^{i\phi(y)}\langle x, y\rangle$$

for all $y \in X$. Thus, to within a multiplicative phase factor, f_x is unique.

If $A \in \mathscr{E}(\mathscr{A})$, then $S_z \subseteq [A]$ if and only if z is in the closed span $\overline{\operatorname{sp}}\, A$. Moreover, $S_z \subseteq [y] \wedge [A]$ if and only if $z=y \in \overline{\operatorname{sp}}\, A$. It follows that

$$\hat{f}_x(A)(y)=\sum \{f_x(S_z): S_z \subseteq [y] \cap [A]\}$$
$$=\begin{cases} \langle x, y\rangle & \text{if } y \in \overline{\operatorname{sp}} A \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

For $E \in \mathscr{A}$, we then have

$$P_{E,f_x}(A) = \sum_{x \in E} |\hat{f}_x(A)(y)|^2 = \sum \{|\langle x, y \rangle|^2 : y \in E \cap \overline{\mathrm{sp}}\, A\} \tag{3.9}$$

If $A \in \mathscr{E}(F)$, then, by (3.9), $P_{E,f_x}(A) = P_{F,f_x}(A)$ if and only if

$$\sum \{|\langle x, y \rangle|^2 : y \in E \cap \overline{\mathrm{sp}}\, A\} = \sum_{y \in A} |\langle x, y \rangle|^2 \tag{3.10}$$

Suppose E does not interfere with F relative to f_x. Then, letting $A = \{z\}$ in (3.10) with $z \in F$, we conclude that if $\langle x, z \rangle \neq 0$, then $z \in E$. It follows that E does not interfere with F relative to f_x if and only if $S_x \cap F \subseteq E$. The exchange condition then implies that F does not interfere with E relative to f_x. If $S_x \cap F = F$ (when $\mathscr{H}$ is separable there are many x satisfying this), then $E = F$.

In traditional Hilbert space quantum mechanics, events are usually described by orthogonal projections. Such a description can also be given in the present framework. If M is an orthogonal projection, let $A \subseteq X$ be an orthonormal basis for the range of M. Then $A \in \mathscr{E}(\mathscr{A})$. Although A is not unique, as we shall see, the amplitude and probability formulas are independent of the A that is chosen. Conversely, if $A \in \mathscr{E}(\mathscr{A})$, then there is a unique orthogonal projection M_A whose range is $\overline{\mathrm{sp}}\, A$. For an orthogonal projection M and a corresponding A, we define the amplitude $\hat{f}_x(M)(y) = \hat{f}_x(A)(y)$. Then, by (3.8),

$$\hat{f}_x(M)(y) = \begin{cases} \langle x, y \rangle & \text{if} \quad My = y \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

and (3.11) is independent of the choice of A. Moreover, defining $P_{E,f_x}(M) = P_{E,f_x}(A)$, we have from (3.9)

$$P_{E,f_x}(M) = \sum \{|\langle x, y \rangle|^2 : y \in E,\ My = y\} \tag{3.12}$$

Again, (3.12) is independent of the chosen A. If M is a 1-dimensional projection, then there exists a unique $y \in X$ such that $My = y$. Then (3.7) gives $\hat{f}_x(M) = \langle x, y \rangle$. In general, P_{E,f_x} is not a probability measure on the lattice of orthogonal projections, although $0 \leq P_{E,f_x}(M) \leq 1$ and $P_{E,f_x}(I) = 1$. However, if there exists an $A \in \mathscr{E}(E)$ corresponding to M, we then have

$$P_{E,f_x}(M) = \sum_{y \in A} |\langle x, y \rangle|^2 = \langle Mx, x \rangle \tag{3.13}$$

Denoting the Boolean σ-algebra of orthogonal projections with this property by $\hat{\mathscr{E}}(E)$, we conclude from (3.13) that P_{E,f_x} is a probability measure on $\hat{\mathscr{E}}(E)$. Of course, (3.13) is the usual quantum mechanical formula.

Having noticed the correspondence between events and projections, we see that tests correspond to the usual quantum observables. For example, suppose an observable $\mathcal{O}$ corresponds to a self-adjoint operator T with pure point spectrum $\{\lambda_1, \lambda_2, \ldots\}$. Then by the spectral theorem, we have $T = \sum \lambda_i P_i$, where $\{P_i: i = 1, 2, \ldots\}$ is a unique set of mutually orthogonal projections such that $\sum P_i = I$. Let E be a test such that $P_i x = x$ or $P_i x = 0$ for all $x \in E$, $i = 1, 2, \ldots$. Then the range of each P_i is an event for E. Now each time $\mathcal{O}$ is measured, one of the λ_i is obtained. Since each λ_i corresponds to a unique P_i, we can identify the outcomes of $\mathcal{O}$ with events of E. If the eigenvalues of T are nondegenerate, we can identify the outcomes of $\mathcal{O}$ with the outcomes of E. ■

This last example shows that entities give a generalization of traditional Hilbert space quantum mechanics. For a discussion of the entity generalization of classical mechanics, see Bennett and Foulis (to appear). An entity $X = (X, \mathscr{A}, \Sigma)$ is an *H-entity* if for every $x \in X$ there exists a unique $S_x \in \Sigma$ such that $S_x \subseteq [x]$. An H-entity is *injective* if the map $x \mapsto S_x$ is injective. Notice that the Hilbertian entities of Example 3.2 are injective H-entities. Examples 2.1, 2.3, and 2.4 are not H-entities. Example 2.5 is an H-entity that is not injective. Example 3.1 is not an H-entity; however, if we replace Σ by $\Sigma' = \{S, V\}$, then we obtain a noninjective H-entity. Notice that an H-entity X has a rich supply of states (X must be unital), yet the states are limited, since there is only one state that guarantees each outcome. In general, if $(X, \mathscr{A}, \Sigma)$ is unital, there may not exist a $\Sigma' \subseteq \Sigma$ such that $(X, \mathscr{A}, \Sigma')$ is an H-entity.

Example 3.3. Let $X = (X, \mathscr{A}, \Sigma)$ be the entity with $X = \{x_1, x_2, x_3\}$, $\mathscr{A} = \{E, F\}$ (where $E = \{x_1, x_2\}$ and $F = \{x_3\}$), and $\Sigma = \{S, T\}$ (where $S = \{x_1, x_3\}$ and $T = \{x_2, x_3\}$). Then X is unital. However, X is not an H-entity, since S, T both guarantee x_3. We cannot delete S or T from Σ, since this would no longer give an entity. (Even if it were, it would not be unital and hence not an H-entity.) ■

For $f \in \mathscr{D}(X)$, $E \in \mathscr{A}$, define

$$E_f = \{x \in E: \hat{f}(x) \neq 0\}$$

The next result characterizes noninterference for H-entities.

Theorem 3.1. If X is an H-entity, then E does not interfere with F relative to f if and only if there exists a bijection $\phi: F_f \to E_f$ such that $S_y = S_{\phi(y)}$ for every $y \in F_f$.

Proof. Suppose there exists a bijection ϕ with the given properties and let $\psi=\phi^{-1}$. Then for every $x\in E_f$ we have

$$\hat{f}(x)=f(S_x)=f(S_{\psi(x)})=\hat{f}(\psi(x))$$

For $A\subseteq F$ we have

$$\hat{f}(A)(x)=\sum\{f(S):S\subseteq[x]\wedge[A]\}=\begin{cases}f(S_x) & \text{if } S_x\cap F\subseteq A\\ 0 & \text{otherwise}\end{cases}$$

Hence,

$$\begin{aligned}P_{E,f}(A)&=\sum_{x\in E}|\hat{f}(A)(x)|^2=\sum\{|f(S_x)|^2:x\in E,\,S_x\cap F\subseteq A\}\\&=\sum\{|\hat{f}(x)|^2:x\in E_f,\,S_x\cap F\subseteq A\}\\&=\sum\{|\hat{f}(\psi(x))|^2:\psi(x)\in F_f,\,S_{\psi(x)}\cap F\subseteq A\}\\&=\sum\{|\hat{f}(y)|^2:y\in F_f,\,S_y\cap F\subseteq A\}\\&=\sum\{|\hat{f}(y)|^2:y\in F_f\cap A\}=\sum_{y\in A}|\hat{f}(y)|^2=P_{F,f}(A)\end{aligned}$$

We conclude that E does not interfere with F relative to f.

Conversely, suppose E does not interfere with F relative to f. Then for every $y\in F$ we have

$$\begin{aligned}\sum\{|\hat{f}(x)|^2:x\in E,\,S_x=S_y\}&=\sum\{|f(S_x)|^2:x\in E,\,S_x=S_y\}\\&=\sum_{x\in E}\sum\{|f(S)|^2:S\subseteq[x]\wedge[y]\}\\&=\sum_{x\in E}|\hat{f}(\{y\})(x)|^2=P_{E,f}(\{y\})\\&=P_{F,f}(\{y\})=|\hat{f}(y)|^2\end{aligned}$$

Thus, if $\hat{f}(y)\neq 0$, there exists $x\in E$ such that $S_x=S_y$. This x is unique, since if $x'\in E$ satisfies $S_{x'}=S_y$, then $S_{x'}=S_x$, which implies $x'=x$. Let $\phi(y)$ be the unique $x\in E$ such that $S_x=S_y$. Then ϕ is a map from F_f into E. Now $\phi\colon F_f\to E$ is injective, since $\phi(y)=\phi(y')$ implies the existence of an $x\in E$ such that $S_y=S_x=S_{y'}$. Moreover, $\phi(y)\in E_f$ for $y\in F_f$ since

$$\hat{f}(\phi(y))=f(S_{\phi(y)})=f(S_y)=\hat{f}(y)\neq 0$$

We now show that $\phi: F_f \to E_f$ is surjective. Suppose there exists an $x' \in E_f$ such that $x' \notin \phi(E_f)$. Then

$$1 = \sum_{y \in F_f} |\hat{f}(y)|^2 = \sum \{|\hat{f}(\phi(y))|^2 : \phi(y) \in \phi(F_f)\}$$

$$= \sum_{x \in \phi(F_f)} |\hat{f}(x)|^2 < \sum_{x \in E_f} |\hat{f}(x)|^2 = 1$$

This is a contradiction. Hence, $\phi: F_f \to E_f$ is bijective. ■

Corollary 3.2. In an H-entity, E does not interfere with F relative to f if and only if F does not interfere with E relative to f.

Corollary 3.3. In an injective H-entity, E does not interfere with F relative to f if and only if $E_f = F_f$.

Proof. If $E_f = F_f$, then letting $\phi: F_f \to E_f$ be the identity function, we conclude from Theorem 3.1 that E does not interfere with F relative to f. Conversely, if E does not interfere with F relative to f, then applying Theorem 3.1, we see that there exists a bijection $\phi: F_f \to E_f$ such that $S_y = S_{\phi(y)}$ for all $y \in E_f$. Since X is injective, $\phi(y) = y$, so ϕ is the identity map and $E_f = F_f$. ■

4. COMBINATION OF ENTITIES

This section considers various ways in which entities can be combined and studies their amplitude spaces. Let X_1, X_2 be nonempty sets and let $\mathscr{A}_1 = \{A_\delta : \delta \in \Delta\}$, $\mathscr{A}_2 = \{B_\gamma : \gamma \in \Gamma\}$ be collections of subsets of X_1, X_2, respectively. We denote the disjoint union of X_1 and X_2 by $X_1 \uplus X_2$. We use the notation

$$\mathscr{A}_1 \vee \mathscr{A}_2 = \{A_\delta, B_\gamma : \delta \in \Delta, \gamma \in \Gamma\}$$

$$\mathscr{A}_1 \wedge \mathscr{A}_2 = \{A_\delta \uplus B_\gamma : \delta \in \Delta, \gamma \in \Gamma\}$$

Of course, $\mathscr{A}_1 \vee \mathscr{A}_2$ and $\mathscr{A}_1 \wedge \mathscr{A}_2$ are collections of subsets of $X_1 \uplus X_2$. Now suppose $X_1 = (X_1, \mathscr{A}_1, \Sigma_1)$ and $X_2 = (X_2, \mathscr{A}_2, \Sigma_2)$ are entities. We define the *horizontal sum* of X_1, X_2 by

$$X_1 + X_2 = (X_1 \uplus X_2, \mathscr{A}_1 \vee \mathscr{A}_2, \Sigma_1 \wedge \Sigma_2)$$

and the *direct sum* of X_1, X_2 by

$$X_1 \oplus X_2 = (X_1 \cup X_2, \mathscr{A}_1 \wedge \mathscr{A}_2, \Sigma_1 \vee \Sigma_2)$$

It is straightforward to check that $X_1 + X_2$ and $X_1 \oplus X_2$ are entities. Extensions to horizontal and direct sum with an arbitrary number of summands is routine. An entity is *classical* if it has only one test. An entity is *semi-classical* if it is a horizontal sum of classical entities.

Example 4.1 (Spin Chain). Let $X_1 = (X_1, \mathscr{A}_1, \Sigma_1)$ be the entity with $X_1 = \{u, d\}$, $\mathscr{A}_1 = \{\{u, d\}\}$, and $\Sigma_1 = \{\{u\}, \{d\}\}$. The classical entity X_1 describes a single spin-1/2 measurement, where u stands for up and d for down. Suppose a spin-1/2 particle initially has spin up and we then perform spin measurements at one time unit and at two time units. This can be described by the semiclassical entity $X = X_0 + X_1 + X_1$, where X_0 is the singular entity $X_0 = \{u\}$. We can then write $X = (X, \mathscr{A}, \Sigma)$, where

$$X = \{u_0, u_1, d_1, u_2, d_2\}$$

$$\mathscr{A} = \{E_0, E_1, E_2\}, \qquad E_0 = \{u_0\}, \qquad E_1 = \{u_1, d_1\}, \qquad E_2 = \{u_2, d_2\}$$

$$\Sigma = \{\{u_0, u_1, u_2\}, \{u_0, u_1, d_2\}, \{u_0, d_1, u_2\}, \{u_0, d_1, d_2\}\}$$

Thus, the subscript designates the time at which the measurement is made. If $S \in \Sigma$, let $n(S)$ be the number of successive spin changes. For example, $n(\{u_0, u_1, d_2\}) = 1$, $n(\{u_0, d_1, u_2\}) = 2$. Define $f: \Sigma \to \mathbb{C}$ by $f(S) = i^{n(S)}/2$, $i = \sqrt{-1}$. Then

$$\hat{f}(u_0) = \frac{1}{2} \sum_{S \in \Sigma} i^{n(s)} = \frac{1}{2}(1 + 2i + i^2) = i$$

$$\hat{f}(u_1) = \frac{1}{2} \sum \{i^{n(S)} : S \subseteq [u_1]\} = \frac{1}{2}(1 + i)$$

$$\hat{f}(d_1) = \frac{1}{2}(i + i^2) = \frac{1}{2}(-1 + i)$$

$$\hat{f}(u_2) = \frac{1}{2}(1 + i^2) = 0$$

$$\hat{f}(d_2) = \frac{1}{2}(2i) = i$$

It follows that $f\in\mathscr{D}(X)$. We now consider interference between E_1 and E_2. We have

$$\hat{f}(\{u_2\})(u_1)=f(\{u_0, u_1, u_2\})=\frac{1}{2}$$

$$\hat{f}(\{u_2\})(d_1)=f(\{u_0, d_1, u_2\})=-\frac{1}{2}$$

$$\hat{f}(\{d_2\})(u_1)=f(\{u_0, u_1, d_2\})=\frac{i}{2}$$

$$\hat{f}(\{d_2\})(d_1)=f(\{u_0, d_1, d_2\})=\frac{i}{2}$$

Hence,

$$P_{E_1,f}(\{u_2\})=\frac{1}{2}\neq 0=P_{E_2,f}(\{u_2\})$$

$$P_{E_1,f}(\{d_2\})=\frac{1}{2}\neq 1=P_{E_2,f}(\{d_2\})$$

so a spin test at time 1 interferes with a spin test at time 2. Moreover

$$\hat{f}(\{u_1\})(u_2)=\frac{1}{2}, \qquad \hat{f}(\{u_1\})(d_2)=\frac{i}{2}$$

$$\hat{f}(\{d_1\})(u_2)=-\frac{1}{2}, \qquad \hat{f}(\{d_1\})(d_2)=\frac{i}{2}$$

Hence,

$$P_{E_2,f}(\{u_1\})=\frac{1}{2}=P_{E_1,f}(\{u_1\})$$

$$P_{E_2,f}(\{d_1\})=\frac{1}{2}=P_{E_1,f}(\{d_1\})$$

so a spin test at time 2 does not interfere with a spin test at time 1. This shows that noninterference is not a symmetric relation.

One can make a similar analysis for longer spin-1/2 chains. If measurements are performed at times $0, 1, 2, \ldots, m$, we construct the entity $X=X_0+X_1+\cdots+X_1$, where X_1 is repeated m times. In this case, we define

$f(S)=i^{n(S)}/2^{m/2}$. It can then be shown that $f\in\mathscr{D}(X)$. Notice that in all these cases

$$\left|\sum_{S\in\Sigma} f(S)\right|=|\hat{f}(u_0)|=1 \tag{4.1}$$

One can also describe other repeated measurements such as higher spin measurements in this way. Of course, in such cases the definition of f would be more complicated. ∎

We now show how to combine amplitudes of the summands to obtain amplitudes for the horizontal and direct sums. First consider a horizontal sum $X=X_1+X_2$. Suppose $f_1\in\mathscr{H}(X_1)$, $f_2\in\mathscr{H}(X_2)$ satisfy

(a) $\sum_{S_1\in\Sigma_1} f_1(S_1)$, $\sum_{S_2\in\Sigma_2} f_2(S_2)$ converge absolutely

(b) $\left|\sum_{S_1\in\Sigma_1} f_1(S_1)\right|=\left|\sum_{S_2\in\Sigma_2} f_2(S_2)\right|=c\neq 0$

(c) $\|f_1\|=\|f_2\|\neq 0$

Letting $c_1=\sum_{S_1\in\Sigma_1} f_1(S_1)$ and $c_2=\sum_{S_2\in\Sigma_2} f_2(S_2)$, we have $|c_1|=|c_2|=c$. Then define $f_1\circ f_2\colon \Sigma_1\wedge\Sigma_2\to\mathbb{C}$ by

$$(f_1\circ f_2)(S_1\cup S_2)=f_1(S_1)f_2(S_2)/c$$

Let $x\in X$ and suppose $x\in X_1$. Then

$$\begin{aligned}(f_1\circ f_2)^\wedge(x)&=\sum\{f_1\circ f_2(S_1\cup S_2)\colon S_1\cup S_2\subseteq[x]\}\\ &=\frac{1}{c}\sum\{f_1(S_1)f_2(S_2)\colon S_1\subseteq[x],\ S_2\in\Sigma_2\}\\ &=\frac{1}{c}\sum_{S_2\in\Sigma_2} f_2(S_2)\sum_{S_1\subseteq[x]} f_1(S_1)=\frac{c_2}{c}\hat{f}_1(x)\end{aligned}$$

For any $E_1\in\mathscr{A}_1$, we have

$$\sum_{x\in E_1}|\{f_1\circ f_2)^\wedge(x)|^2=\sum_{x\in E_1}|\hat{f}_1(x)|^2=\|f_1\|^2$$

Similarly, if $x\in X_2$, then $(f_1\circ f_2)^\wedge(x)=c_1\hat{f}_2(x)/c$ and for any $E_2\in\mathscr{A}_2$, we have

$$\sum_{x\in E_2}|(f_1\circ f_2)^\wedge(x)|^2=\sum_{x\in E_2}|\hat{f}_2(x)|^2=\|f_2\|^2$$

we conclude that $f_1\circ f_2\in\mathscr{H}(X)$ and that $\|f_1\circ f_2\|=\|f_1\|=\|f_2\|$. Moreover, if $f_1\in\mathscr{D}(X_1)$, $f_2\in\mathscr{D}(X_2)$, then $f_1\circ f_2\in\mathscr{D}(X)$.

It is certainly true that (b) and (c) are strong conditions that greatly restrict the amplitudes that can be combined in this manner. However, there are amplitudes that satisfy these conditions. For example, suppose $X_1 = X_2$ and $f_1 = f_2$ satisfy (a). As another example, suppose X_i, $f_i \in \mathscr{D}(X_i)$, $i = 1, 2$, describe spin chains as in Example 4.1. Then (a) is satisfied, $\|f_1\| = \|f_2\| = 1$, and, by (4.1), we see that (b) is satisfied with $c = 1$.

We next study independence and interference in $X = X_1 + X_2$. Suppose $f_1 \in \mathscr{D}(X_1)$ and $f_2 \in \mathscr{D}(X_2)$ satisfy (a) and (b) with $c = 1$. We have shown that $f = f_1 \circ f_2 \in \mathscr{D}(X)$. Then for $A \in \mathscr{E}(\mathscr{A}_2)$ and $x \in X_1$, we have

$$\begin{aligned} \hat{f}(A)(x) &= \sum \{f(S_1 \cup S_2) : S_1 \cup S_2 \subseteq [x] \wedge [A]\} \\ &= \sum \{f_1(S_1) f_2(S_2) : S_1 \subseteq [x], S_2 \subseteq [A]\} \\ &= \sum \{f_1(S_1) : S_1 \subseteq [x]\} \sum \{f_2(S_2) : S_2 \subseteq [A[\} \\ &= \hat{f}_1(x) \sum \{f_2(S_2) : S_2 \subseteq [A]\} \end{aligned} \tag{4.2}$$

If $E \in \mathscr{A}_1$, then applying (4.2) gives

$$P_{E,f}(A) = \sum_{x \in E} |\hat{f}(A)(x)|^2 = \left| \sum \{f_2(S_2) : S_2 \subseteq [A]\} \right| \tag{4.3}$$

Now let $B \subseteq E$ and $A \subseteq F \in \mathscr{A}_2$. Applying (4.2) and (4.3), we have

$$\begin{aligned} \sum_{x \in B} |\hat{f}(A)(x)|^2 &= \sum_{x \in B} |\hat{f}_1(x)|^2 \left| \sum \{f_2(S_2) : S_2 \subseteq [A]\} \right|^2 \\ &= P_{E,f}(B) P_{E,f}(A) \end{aligned}$$

We conclude from (3.6) that E and F are independent relative to f. Hence, any test in $\mathscr{A}_1$ is independent of any test in $\mathscr{A}_2$ (and conversely) relative to f. This is not surprising, since f has the form $f = f_1 \circ f_2$.

We now consider interference. It follows from (4.3) that $P_{E,f}(A) = P_{G,f}(A)$ for every E, $G \in \mathscr{A}_1$, $A \in \mathscr{E}(\mathscr{A}_2)$. Again, for $A \subseteq F \in \mathscr{A}_2$, it easily follows that

$$P_{F,f}(A) = \sum_{x \in F} |\hat{f}(A)(x)|^2 = \sum_{x \in A} |\hat{f}_2(x)|^2 \tag{4.4}$$

Now (4.3) and (4.4) certainly look different. In fact, we shall see in Example 4.2 that there can exist an $f_2 \in \mathscr{D}(X_2)$ such that $P_{E,f}(A) \neq P_{F,f}(A)$. Hence, for such an f, every $E \in \mathscr{A}_1$ interferes with every $F \in \mathscr{A}_2$. This is related to the EPR problem. The systems X_1 and X_2 are separated in the sense that the tests in X_1 cannot communicate with the tests in X_2. However, the states can "communicate," so we have interference of tests. Thus, we have nonlocality. This also shows that there are independent tests that interfere. Combining this observation with Example 3.1 shows that there is no relationship between independence and noninterference. That is, each of the four combinations of independence/dependence and interference/noninterference is possible.

Example 4.2. Let $(X_2, \mathscr{A}_2, \Sigma_2)$ be the classical entity with $X_2 = \{x_1, x_2, x_3\}$, $\mathscr{A}_2 = \{X_2\}$, and $\Sigma_2 = \{S, T, U\}$, where $S = \{x_1\}$, $T = \{x_2\}$, and $U = \{x_3\}$. Define $f_2 : \Sigma_2 \to \mathbb{C}$ by $f_2(S) = 1/3 + i\sqrt{3}$, $f_2(T) = 1/3 - i\sqrt{3}$, and $f_2(U) = 1/3$. Then $\hat{f}_2(x_1) = f_2(S)$, $\hat{f}_2(x_2) = f_2(T)$, and $\hat{f}_2(x_3) = f(U)$, so $f_2 \in \mathscr{D}(X_2)$. Letting $A = \{x_1, x_2\} \in \mathscr{E}(\mathscr{A}_2)$, we have

$$|\sum \{f_2(S_2) : S_2 \subseteq [A]\}|^2 = \left|\frac{1}{3} + \frac{i}{\sqrt{3}} + \frac{1}{3} - \frac{i}{\sqrt{3}}\right|^2 = \frac{4}{9} \neq \frac{8}{9}$$

$$= \left|\frac{1}{3} + \frac{i}{\sqrt{3}}\right|^2 + \left|\frac{1}{3} - \frac{i}{\sqrt{3}}\right|^2 = \sum_{x \in A} |\hat{f}_2(x)|^2$$

We conclude that (4.3) and (4.4) do not agree, in general. ■

Let $X = X_1 + X_2$. We now characterize those $f \in \mathscr{H}(X)$ that have the form $f = f_1 \circ f_2$, $f_1 \in \mathscr{H}(X_1)$, and $f_2 \in \mathscr{H}(X_2)$. We say that $f \in \mathscr{H}(X)$ is *factorizable* if

(d) $\sum_{S \in \Sigma} f(S) = d \neq 0$ converges absolutely

(e) if $f'(S_1) = \sum_{S_2 \in \Sigma_2} f(S_1 \cup S_2)$, $S_1 \in \Sigma_1$,

then $f'(S_1) = 0$ implies $f(S_1 \cup S_2) = 0$ for all $S_2 \in \Sigma_2$

(f) $f(S_1 \cup S_2)/f'(S_1)$ depends only on S_2 whenever $f'(S_1) \neq 0$

Theorem 4.1. Let $f \in \mathscr{H}(X)$ with $\|f\| \neq 0$. Then there exist $f_1 \in \mathscr{H}(X_1)$ and $f_2 \in \mathscr{H}(X_2)$ such that $f = f_1 \circ f_2$ if and only if f is factorizable.

Proof. Suppose $f = f_1 \circ f_2$. Then f_1, f_2 satisfy (a)–(c) and $f(S_1 \cup S_2) = f_1(S_1) f_2(S_2)$. We then have

$$\sum_{S \in \Sigma} f(S) = \sum_{S_1 \in \Sigma_1} f_1(S_1) \sum_{S_2 \in \Sigma_2} f_2(S_2) = c_1 c_2 \neq 0$$

and the series converges absolutely, so (d) holds. Since $f'(S_1) = c_2 f_1(S_1)$, if $f'(S_1) = 0$, then $f_1(S_1) = 0$, so $f(S_1 \cup S_2) = 0$ for every $S_2 \in \Sigma_2$. Hence, (e) holds. If $f'(S_1) \neq 0$, then

$$\frac{f(S_1 \cup S_2)}{f'(S_1)} = \frac{1}{c_2} f_2(S_2)$$

is a function of only S_2, so (f) holds. Conversely, suppose $f\in\mathscr{H}(X)$ is factorizable. We first show that $f'\in\mathscr{H}(X_1)$. For $x\in X_1$, we have

$$\hat{f}'(x)=\sum_{S_1\subseteq[x]} f'(S_1)=\sum_{S_1\subseteq[x]}\sum_{S_2\subseteq\Sigma_2} f(S_1\cup S_2)$$
$$=\sum\{f(S_1\cup S_2): S_1\cup S_2\subseteq[x]\}=\hat{f}(x) \tag{4.5}$$

Hence, for $E, F\in\mathscr{A}_1$, we have

$$\sum_{x\in E}|\hat{f}'(x)|^2=\sum_{x\in E}|\hat{f}(x)|^2=\sum_{x\in F}|\hat{f}(x)|^2=\sum_{x\in F}|\hat{f}'(x)|^2$$

Since $\|f\|\neq 0$, if follows from (4.5) that there exists an $S_1\in\Sigma_1$ such that $f'(S_1)\neq 0$. Define

$$f''(S_2)=\frac{f(S_1\cup S_2)}{f'(S_1)}$$

for all $S_2\in\Sigma_2$. By (e),

$$f'(S_1)f''(S_2)=f(S_1\cup S_2) \tag{4.6}$$

for all $S_1\in\Sigma_1$, $S_2\in\Sigma_2$. Applying (d) and (e), we have

$$df''(S_2)=\sum_{S_1\in\Sigma_1} f(S_1\cup S_2)$$

As in (4.5), if $x\in X_2$, then $\hat{f}''(x)=\hat{f}(x)/d$, so $f''\in\mathscr{H}(X_2)$. It follows from (4.2) that $f=f'\circ f''$. ■

Of course, we could interchange the roles of Σ_1 and Σ_2 in the definition of factorizability.

We now consider the direct sum $X=X_1\oplus X_2$. Let $f_1\in\mathscr{H}(X_1)$ and $f_2\in\mathscr{H}(X_2)$ and define $f=f_1\oplus f_2\colon\Sigma_1\vee\Sigma_2\to\mathbb{C}$ by

$$f(S)=\begin{cases} f_1(S) & \text{if } S\in\Sigma_1\\ f_2(S) & \text{if } S\in\Sigma_2\end{cases}$$

Let $x\in X$ and suppose $x\in X_1$. Then

$$\hat{f}(x)=\sum_{S\subseteq[x]} f(S)=\sum\{f_1(S_1): S_1\in\Sigma_1, S_1\subseteq[x]\}=\hat{f}_1(x)$$

Similarly, if $x\in X_2$, $\hat{f}(x)=\hat{f}_2(x)$. Hence, for $E=E_1\cup E_2\in\mathscr{A}_1\wedge\mathscr{A}_2$ we have

$$\sum_{x\in E}|\hat{f}(x)|^2=\sum_{x\in E_1}|\hat{f}(x)|^2+\sum_{x\in E_2}|\hat{f}(x)|^2$$
$$=\sum_{x\in E_1}|\hat{f}_1(x)|^2+\sum_{x\in E_2}|\hat{f}_2(x)|^2=\|f_1\|^2+\|f_2\|^2$$

We conclude that $f \in \mathscr{H}(X)$ and $\| f \|^2 = \| f_1 \|^2 + \| f_2 \|^2$. Moreover, if $f_1 \in \mathscr{D}(X_1)$, $f_2 \in \mathscr{D}(X_2)$, and $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$, then $af_1 \oplus bf_2 \in \mathscr{D}(X)$.

We now discuss interference and independence. Let

$$A = A_1 \cup A_2 \in \mathscr{E}(\mathscr{A}_1 \wedge \mathscr{A}_2)$$

$f = af_1 \oplus bf_2$, $f_i \in \mathscr{D}(X_i)$, $i = 1, 2$, $|a|^2 + |b|^2 = 1$. If $x \in X_1$, then

$$\begin{aligned} \hat{f}(A)(x) &= \sum \{ f(S) : S \subseteq [x] \wedge [A] \} \\ &= \sum \{ af_1(S_1) : S_1 \in \Sigma_1, S_1 \subseteq [x] \wedge [A_1] \} \\ &= a\hat{f}_1(A_1)(x) \end{aligned}$$

Similarly, if $x \in X_2$, then $\hat{f}(A)(X) = b\hat{f}_2(A_2)(x)$. For $E = E_1 \cup E_2 \in \mathscr{A}_1 \wedge \mathscr{A}_2$, we have

$$\begin{aligned} P_{E,f}(A) &= \sum_{x \in E} |\hat{f}(A)(x)|^2 \\ &= |a|^2 \sum_{x \in E_1} |\hat{f}_1(A_1)(x)|^2 + |b|^2 \sum_{x \in E_2} |\hat{f}_2(A_2)(x)|^2 \\ &= |a|^2 P_{E_1, f_1}(A_1) + |b|^2 P_{E_2, f_2}(A_2) \end{aligned} \tag{4.7}$$

This shows that the distribution of E relative to f is a convex combination of the distribution of E_1 relative to f_1 and that of E_2 relative to f_2. Let $F = F_1 \cup F_2 \in \mathscr{A}_1 \wedge \mathscr{A}_2$. It follows from (4.7) that if E_i does not interfere with F_i relative to f_i, $i = 1, 2$, then E does not interfere with F relative to f. Let $B = B_1 \cup B_2 \in \mathscr{E}(E)$, $A \in \mathscr{E}(F)$. Then

$$\sum_{x \in B} |\hat{f}(A)(x)|^2 = |a|^2 \sum_{x \in B_1} |\hat{f}_1(A_1)(x)|^2 + |b|^2 \sum_{x \in B_2} |\hat{f}_2(A_2)(x)|^2 \tag{4.8}$$

while

$$\begin{aligned} P_{E,f}(B) P_{E,f}(A) = {} & [|a|^2 P_{E_1, f_1}(B_1) + |b|^2 P_{E_2, f_2}(B_2)] \\ & \times [|a|^2 P_{E_1, f_1}(A_1) + |b|^2 P_{E_2, f_2}(A_2)] \end{aligned} \tag{4.9}$$

If E_i is independent of F_i relative to f_i, $i = 1, 2$, then (4.8) gives

$$\sum_{x \in B} |\hat{f}(A)(x)|^2 = |a|^2 P_{E_1, f_1}(A_1) P_{E_1, f_1}(B_1) + |b|^2 P_{E_2, f_2}(A_2) P_{E_2, f_2}(B_2) \tag{4.10}$$

In general, (4.9) and (4.10) do not coincide unless $ab = 0$. Thus, even in the case of componentwise independence, E and F are not independent in general.

The next result characterizes amplitude density direct sums.

Theorem 4.2. Let $X=X_1 \oplus X_2$ and $f \in \mathscr{D}(X)$ and assume $\mathscr{D}(X_1)$, $\mathscr{D}(X_2) \neq \varnothing$. Then there exist $f_i \in \mathscr{D}(X_i)$, $i=1,2$, and $a, b \in \mathbb{C}$ with $|a|^2+|b|^2=1$ such that $f=af_1 \oplus bf_2$ if and only if (1) $\hat{f}(x)=0$ for every $x \in X_1$ implies $f(S_1)=0$ for every $S_1 \in \Sigma_1$, and (2) $\hat{f}(x)=0$ for every $x \in X_2$ implies $f(S_2)=0$ for every $S_2 \in \Sigma_2$.

Proof. Suppose $f \in \mathscr{D}(X)$ and $f=af_1 \oplus bf_2$, $f_i \in \mathscr{D}(X_i)$, $i=1,2$. Assume $\hat{f}(x)=0$ for every $x \in X_1$. Then for every $x \in X_1$, $a\hat{f}_1(x)=\hat{f}(x)=0$. Since there exists an $x \in X$ such that $\hat{f}_1(x) \neq 0$, $a=0$. Hence, $|b|=1$ and $f=0f_1 \oplus bf_2$. Then for every $S_1 \in \Sigma_1$ we have $f(S_1)=0f_1(S_1)=0$. Therefore, (1) holds and in a similar way (2) holds. Conversely, suppose $f \in \mathscr{D}(X)$ and (1) and (2) hold. Now for $E=E_1 \cup E_2 \in a_1 \wedge a_2$ we have

$$1=\sum_{x \in E} |\hat{f}(x)|^2=\sum_{x \in E_1} |\hat{f}(x)|^2+\sum_{x \in E_2} |\hat{f}(x)|^2$$

Define $a \geq 0$ by

$$a^2=1-\sum_{x \in E_2} |\hat{f}(x)|^2$$

for a fixed $E_2 \in \mathscr{A}_2$. Then for every E_1, $E_1' \in \mathscr{A}_1$ we have

$$\sum_{x \in E_1} |\hat{f}(x)|^2=\sum_{x \in E_1'} |\hat{f}(x)|^2=a^2$$

Similarly, there is a $b \geq 0$ such that for any E_2, $E_2' \in \mathscr{A}_2$ we have

$$\sum_{x \in E_2} |\hat{f}(x)|^2=\sum_{x \in E_2'} |\hat{f}(x)|^2=b^2$$

Then $a^2+b^2=1$. If $a=0$, then $\hat{f}(x)=0$ for every $x \in X_1$, so $f(S_1)=0$ for every $S_1 \in \Sigma_1$. Let $f_1 \in \mathscr{D}(X_1)$ be arbitrary and define $f_2\colon \Sigma_2 \to \mathbb{C}$ by $f_2(S_2)=f(S_2)$. To show that $f_2 \in \mathscr{D}(X_2)$, let $x \in X_2$. Then

$$\hat{f}_2(x)=\sum_{S_2 \subseteq [x]} f_2(S_2)=\sum_{S_2 \subseteq [x]} f(S_2)=\sum_{S \subseteq [x]} f(S)=\hat{f}(x)$$

Hence, if $E_2 \in \mathscr{A}_2$, we have

$$\sum_{x \in E_2} |\hat{f}_2(x)|^2=\sum_{x \in E_2} |\hat{f}(x)|^2=b^2=1$$

Therefore, $f_2 \in \mathscr{D}(X_2)$ and $f = 0f_1 \oplus f_2$. Similarly, if $b=0$, then $f = f_1 \oplus 0f_2$ for some $f_i \in \mathscr{D}(X_i)$, $i=1, 2$. Now suppose $a, b > 0$. Define $f_1 : \Sigma_1 \to \mathbb{C}$ by $f_1(S_1) = f(S_1)/a$ and $f_2 : \Sigma_2 \to \mathbb{C}$ by $f_2(S_2) = f(S_2)/b$. To show that $f_1 \in \mathscr{D}(X_1)$, let $x \in X_1$. Then

$$\hat{f}_1(x) = \sum_{S_1 \subseteq [x]} f_1(S_1) = \frac{1}{a} \sum_{S_1 \subseteq [x]} f(S_1) = \frac{1}{a} \sum_{S \subseteq [x]} f(S) = \frac{1}{a} \hat{f}(x)$$

Hence, for $E_1 \in \mathscr{A}_1$ we have

$$\sum_{x \in E_1} |\hat{f}_1(x)|^2 = \frac{1}{a^2} \sum_{x \in E_1} |\hat{f}(x)|^2 = 1$$

Similarly, $f_2 \in \mathscr{D}(X_2)$. Moreover, $f = af_1 \oplus bf_2$. ■

Corollary 4.3. Let $f \in \mathscr{D}(X_1 \oplus X_2)$ and suppose there exist $x_i \in X_i$ such that $\hat{f}(x_i) \neq 0$, $i=1, 2$. Then there exist $f_i \in \mathscr{D}(X_i)$, $i=1, 2$, and $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$ such that $f = af_1 \oplus bf_2$.

Corollary 4.4. Let $f \in \mathscr{H}(X_1 \oplus X_2)$ and suppose there exist $x_i \in X_i$ such that $\hat{f}(x_i) \neq 0$, $i=1, 2$. Then there exist $f_i \in \mathscr{H}(X_i)$, $i=1, 2$, such that $f = f_1 \oplus f_2$.

The next result shows that the decomposition $f = af_1 \oplus bf_2$, for

$$f \in \mathscr{D}(X_1 \oplus X_2)$$

is essentially unique.

Lemma 4.5. Let $f_i, f_i' \in \mathscr{D}(X_i)$, $i=1, 2$, and suppose

$$af_1 \oplus bf_2 = a'f_1' \oplus b'f_2'$$

where $a, b \neq 0$. Then there exist $c, d \in \mathbb{C}$ with $|c| = |d| = 1$ such that $f_1 = cf_1'$, $f_2 = df_2'$, and $a' = ac$, $b' = bd$.

Proof. Since $af_1(S_1) = a'f_1'(S_1)$ for every $S_1 \in \Sigma_1$, we have

$$f_1(S_1) = \frac{a'}{a} f_1'(S_1)$$

for every $S_1 \in \Sigma_1$. Hence, $\hat{f}_1(x) = a'\hat{f}_1'(x)/a$ for all $x \in X_1$. Moreover, for $E_1 \in \mathscr{A}_1$, we have

$$1 = \sum_{x \in E_1} |\hat{f}_1(x)|^2 = \left|\frac{a'}{a}\right|^2 \sum_{x \in E_1} |\hat{f}_1'(x)|^2 = \left|\frac{a'}{a}\right|^2$$

Letting $c = a'/a$, we have $f_1 = cf_1'$ and $a' = ac$. A similar result holds for f_2. ■

We now give an example of an $f \in \mathscr{D}(X_1 \oplus X_2)$ which is not of the form $f = af_1 \oplus bf_2$, $f_i \in \mathscr{D}(X_i)$, $i=1, 2$.

Example 4.3. Let $(X_1, \mathscr{A}_1, \Sigma_1)$ be the semiclassical entity given by $X_1 = \{x_1, x_2, x_3, x_4\}$, $\mathscr{A}_1 = \{E_1, F_1\}$ (where $E_1 = \{x_1, x_2\}$ and $F_1 = \{x_3, x_4\}$, and $\Sigma_1 = \{S_1, S_2, S_3, S_4\}$ (where $S_1 = \{x_1, x_3\}$, $S_1 = \{x_1, x_4\}$, $S_3 = \{x_2, x_4\}$, and $S_4 = \{x_2, x_3\}$). Let X_2 be the singular entity $X_2 = \{x_5\}$, $\mathscr{A}_2 = \{E_2\}$, $\Sigma_2 = \{S_5\}$, and $E_2 = S_5 = \{x_5\}$. Define $f\colon \Sigma_1 \vee \Sigma_2$ by

$$f(S_1) = -f(S_2) = f(S_3) = -f(S_4) = f(S_5) = 1$$

Then $f \in \mathscr{D}(X_1 \oplus X_2)$. Indeed, $\hat{f}(x_5) = 1$ and $\hat{f}(x_i) = 0$, $i = 1, 2, 3, 4$. Hence, for $E = E_1 \cup E_2$ and $F = F_1 \cup F_2$ we have

$$\sum_{x \in E} |\hat{f}(x)|^2 = \sum_{x \in F} |\hat{f}(x)|^2 = 1$$

Now $\hat{f}(x) = 0$ for every $x \in X_1$, yet $f(S) \neq 0$ for $S \in \Sigma_1$. By Theorem 4.2, $f \neq af_1 \oplus bf_2$ for $f_i \in \mathscr{D}(X_i)$, $i = 1, 2$. ■

If A and B are sets, we denote their Cartesian product $A \times B$ by AB. In particular, if $A = \{a\}$ is a singleton set, we write aB for $\{a\}B$ and similarly we write Ab for $A\{b\}$. We denote an element $(a, b) \in AB$ by ab. If $X = (X, \mathscr{A}, \Sigma)$, $Y = (Y, \mathscr{B}, \Lambda)$ are entities, the *Cartesian product* of X and Y is the entity

$$XY = (XY, \mathscr{A}\mathscr{B}, \Sigma\Lambda)$$

where $\mathscr{A}\mathscr{B} = \{EF\colon E \in \mathscr{A}, F \in \mathscr{B}\}$ and $\Sigma\Lambda = \{ST\colon S \in \Sigma, T \in \Lambda\}$. It is easy to check that XY is indeed an entity. For $f \in \mathscr{H}(X)$, $g \in \mathscr{H}(Y)$, we define $fg\colon \Sigma\Lambda \to \mathbb{C}$ by $fg(ST) = f(S)g(T)$. If $xy \in XY$, we have

$$(fg)^\wedge(xy) = \sum_{ST \subseteq [xy]} (fg)(ST) = \sum_{S \subseteq [x]} f(S) \sum_{T \subseteq [y]} f(T) = \hat{f}(x)\hat{g}(y)$$

We then have for any $EF \in \mathscr{A}\mathscr{B}$

$$\sum_{xy \in EF} |(fg)^\wedge(xy)|^2 = \sum_{x \in E} |\hat{f}(x)|^2 \sum_{y \in F} |\hat{f}(y)|^2$$

It follows that $fg \in \mathscr{H}(XY)$ and $\|fg\| = \|f\| \, \|g\|$. In particular, if $f \in \mathscr{D}(X)$, $g \in \mathscr{D}(Y)$, then $fg \in \mathscr{D}(XY)$.

If $C \in \mathscr{E}(\mathscr{A}\mathscr{B})$, then

$$\begin{aligned}(fg)^\wedge(C)(xy) &= \sum \{(fg)(ST)\colon ST \subseteq [xy] \wedge [C]\} \\ &= \sum \{f(S)g(T)\colon S \subseteq [x], T \subseteq [y], ST \subseteq [C]\}\end{aligned}$$

In particular, if $C=AB$, $A\in\mathscr{E}(\mathscr{A})$, $B\in\mathscr{E}(\mathscr{B})$, then

$$\begin{aligned}(fg)^{\wedge}(AB)(xy)&=\sum\{f(S)g(T): S\subseteq[x]\wedge[A],\ T\subseteq[y]\wedge[B]\}\\&=\hat{f}(A)(x)\hat{g}(B)(y)\end{aligned}\tag{4.11}$$

If $EF\in\mathscr{A}\mathscr{B}$, then applying (4.11) gives

$$P_{EF,fg}(AB)=P_{E,f}(A)P_{F,g}(B)\tag{4.12}$$

Suppose E, $E'\in\mathscr{A}$ and F, $F'\in\mathscr{B}$, E does not interfere with E', F does not interfere with F', and $AB\in\mathscr{E}(E'F')$. Then from (4.12) we have

$$P_{EF,fg}(AB)=P_{E',f}(A)P_{F',f}(B)=P_{E'F',fg}(AB)$$

However, every $C\in\mathscr{E}(E'F')$ need not be a product event $C=AB$ and in general we may have

$$P_{EF,fg}(C)\neq P_{E'F',fg}(C)$$

so EF can interfere with $E'F'$ relative to fg. A similar observation holds for independence.

Nevertheless, in a certain sense, an $fg\in\mathscr{D}(XY)$ does not give interference between $E\in\mathscr{A}$ and $F\in\mathscr{B}$, since for $A\subseteq E$,

$$P_{EF,fg}(AF)=P_{E,f}(A)P_{EF,g}(F)=P_{E,f}(A)$$

Thus, the presence of F does not affect E. Similarly, if $B\subseteq F$, then $P_{EF,fg}(EB)=P_{F,g}(B)$, so the presence of E does not affect F. We now give a way of combining amplitudes that does allow interference and which also describes a temporal structure. Let us view XY as an entity in which we first execute a test in X and at a later time we execute a test in Y. [A more delicate temporal description is given by the forward operational produce (Foulis, 1989; Foulis and Randall, 1972*b*).] If $S\in\Sigma$ and $T\in\Lambda$, we can write $ST=\uplus_{t\in T}St$, which is interpreted as meaning that the first system is in state S and later some $t\in T$ is possible. For $f\in\mathscr{D}(X)$ and $g\in\mathscr{D}(Y)$ we want to define a product $\overrightarrow{fg}$ so that $\overrightarrow{fg}\,(ST)$ is the amplitude that the first system is in state S and that, given this fact, the second system is in state T at a later time. Heuristically, if we assume an additivity and multiplicative condition, we might have

$$\overrightarrow{fg}(ST)=\overrightarrow{fg}\left(\biguplus_{t\in T}St\right)=\sum_{t\in T}\overrightarrow{fg}(St)=cf(S)\sum_{t\in T}\hat{g}(t)\tag{4.13}$$

where c is a normalization constant. Of course, the second and third equalities in (4.13) are meaningless in the present context. However, the last expression in (4.13) does have meaning if the summation converges.

To make this motivation rigorous, we say that $f \in \mathscr{D}(X)$ is a *strong amplitude density* if $\tilde{f}(S)=\sum_{s\in S}\hat{f}(s)$ converges absolutely for every $S\in\Sigma$ and $\tilde{f}\in\mathscr{H}(X)$ with $\|\tilde{f}\|\neq 0$. If $f\in\mathscr{D}(X)$ is strong, we define $f'=\tilde{f}/\|\tilde{f}\|$. Then $f'\in\mathscr{D}(X)$. Now suppose $f\in\mathscr{D}(X)$ and $g\in\mathscr{D}(Y)$, where g is strong. We then define $\overrightarrow{fg}\in\mathscr{D}(XY)$ by $\overrightarrow{fg}=fg'$. We then have

$$\overrightarrow{fg}(ST)=f(S)g'(T)=\frac{1}{\|\tilde{g}\|}f(S)\sum_{t\in T}\hat{g}(t)$$

so $\overrightarrow{fg}$ satisfies the last equality in (4.13). If $EF\in\mathscr{A}\mathscr{B}$ and $AB\in\mathscr{E}(\mathscr{A}\mathscr{B})$, then by (4.12) we have

$$P_{EF,\overrightarrow{fg}}(AB)=P_{EF,fg'}(AB)=P_{E,f}(A)P_{F,g'}(B)$$

In particular, if $A\subseteq E$, then

$$P_{EF,\overrightarrow{fg}}(AF)=P_{E,f}(A)$$

Hence, F does not interfere with E in our previous sense. However, if $B\subseteq F$, then

$$P_{EF,\overrightarrow{fg}}(EB)=P_{F,g'}(B)$$

In general, $P_{F,g'}(B)\neq P_{F,g}(B)$, so E interferes with F in this sense.

Although strongness is a restriction on an amplitude density, the next example shows that in certain cases this is no restriction at all.

Example 4.4. Let $(X,\mathscr{A},\Sigma)$ be the firefly entity of Example 3.1. We shall show that every $f\in\mathscr{D}(X)$ is strong. Let $f\in\mathscr{D}(X)$ with $f(S)=a$, $f(T)=b$, $f(U)=c$, and $f(V)=d$. Then, as in Example 3.1, we have

$$|a+c|^2+|b+d|^2=|c+d|^2+|a+b|^2=1 \tag{4.14}$$

Now

$$\begin{aligned}\tilde{f}(S)&=\hat{f}(x_1)+\hat{f}(x_4)=2a+b+c\\ \tilde{f}(T)&=\hat{f}(x_2)+\hat{f}(x_4)=2b+a+d\\ \tilde{f}(U)&=\hat{f}(x_1)+\hat{f}(x_3)=2c+a+d\\ \tilde{f}(V)&=\hat{f}(x_2)+\hat{f}(x_3)=2d+b+c\end{aligned}$$

and

$$\begin{aligned}\tilde{f}^{\wedge}(x_1)&=3(a+c)+b+d\\ \tilde{f}^{\wedge}(x_2)&=3(b+d)+a+c\\ \tilde{f}^{\wedge}(x_3)&=3(c+d)+a+b\\ \tilde{f}^{\wedge}(x_4)&=3(a+b)+c+d\end{aligned}$$

Hence,

$$\|\tilde{f}^\wedge\|_E^2 = |3(a+c)+b+d|^2 + |3(b+d)+a+c|^2$$
$$\|\tilde{f}^\wedge\|_F^2 = |3(c+d)+a+b|^2 + |3(a+b)+c+d|^2 \tag{4.15}$$

Now $\tilde{f} \in \mathscr{H}(X)$ if and only if

$$\|\tilde{f}^\wedge\|_E^2 = \|\tilde{f}^\wedge\|_F^2 \tag{4.16}$$

Applying (4.14) and (4.15) gives

$$\|\tilde{f}^\wedge\|_E^2 = 10 + 12\,\mathrm{Re}(a+c)(\overline{b+d})$$
$$\|\tilde{f}^\wedge\|_F^2 = 10 + 12\,\mathrm{Re}(c+d)(\overline{a+b})$$

Hence, (4.16) holds if and only if

$$\mathrm{Re}(a+c)(\overline{b+d}) = \mathrm{Re}(c+d)(\overline{a+b}) \tag{4.17}$$

But (4.17) is equivalent to

$$\mathrm{Re}(a\bar{b} + c\bar{d}) = \mathrm{Re}(c\bar{a} + d\bar{b}) \tag{4.18}$$

and (4.14) implies that (4.18) holds. Hence, $\tilde{f} \in \mathscr{H}(X)$. We must now show that $\|\tilde{f}\| \neq 0$. If $\|\tilde{f}\| = 0$, then by (4.15) we have

$$3(a+c)+b+d = 3(b+d)+a+c = 3(c+d)+a+b = 3(a+b)+c+d = 0$$

But the only solution of these equations is $a=b=c=d=0$, which contradicts (4.14). ■

We call $f \in \mathscr{H}(XY)$ a *product amplitude* if $f = gh$ for some $g \in \mathscr{H}(X)$ and $h \in \mathscr{H}(Y)$. For $f \in \mathscr{H}(XY)$, $S \in \Sigma$, and $T \in \Lambda$, define $f_T(S) = f(ST)$ and ${}_sf(T) = f(ST)$. The next result characterizes product amplitudes.

Theorem 4.6. Let $f \in \mathscr{H}(XY)$. Then f is a product amplitude if and only if $f_T \in \mathscr{H}(X)$ and ${}_sf \in \mathscr{H}(Y)$ for every $S \in \Sigma$ and $T \in \Lambda$; and for every $S, S' \in \Sigma$ and $T, T' \in \Lambda$ we have

$$f(ST)f(S'T') = f(ST')f(S'T)$$

Proof. Suppose f is a product amplitude and $f = gh$. Then $f_T = h(T)g$ and ${}_sf = g(S)h$ are amplitudes and

$$f(ST)f(S'T') = g(S)h(T)g(S')h(T') = g(S)h(T')g(S')h(T)$$
$$= f(ST')f(S'T)$$

Conversely, suppose f satisfies the conditions of the theorem. If $f=0$, then clearly f is a product amplitude. Otherwise, there exist $S'\in\Sigma$ and $T'\in\Lambda$ such that $f(S'T')\neq 0$. Then for every $S\in\Sigma$ and $T\in\Lambda$ we have

$$f(ST)=\frac{f(ST')f(S'T)}{f(S'T')}=\frac{1}{f(S'T')}f_{T'}(S)_{S'}f(T) \quad \blacksquare$$

Other types of entity products can be defined. For example, there are the forward and backward operational products and the tensor product (Foulis, 1989; Gudder, 1988*a*). However, we have not yet developed a definitive amplitude structure for such products, so we shall not pursue them here.

5. SECTORS

Let $X=(X,\mathscr{A},\Sigma)$ be an entity. We define a relation s on $\mathscr{H}(X)$ by $f\,s\,g$ if for every $E, F\in\mathscr{A}$ we have

$$\sum_{x\in E}\hat{f}(x)\overline{\hat{g}(x)}=\sum_{x\in F}\hat{f}(x)\overline{\hat{g}(x)} \tag{5.1}$$

We can write (5.1) as $\langle\hat{f},\hat{g}\rangle_E=\langle\hat{f},\hat{g}\rangle_F$ and if $f\,s\,g$ we write $\langle f,g\rangle=\langle\hat{f},\hat{g}\rangle_E$, where $E\in\mathscr{A}$ is arbitrary. Notice that if $f\,s\,g$, then $af\,s\,g$ for all $a\in\mathbb{C}$. It is clear that s is a symmetric, reflexive relation and $\langle f,f\rangle=\|f\|^2$. However, as we shall see, s need not be transitive. We call $f\in\mathscr{H}(X)$ a *null amplitude* if $\|f\|=0$ and denote the set of null amplitudes by $\mathscr{N}(X)$. Thus, $f\in\mathscr{N}(X)$ if and only if $\hat{f}\equiv 0$. If $f\in\mathscr{N}(X)$, then clearly $f\,s\,g$ for all $g\in\mathscr{H}(X)$. If $f\in\mathscr{H}(X)$, then of course $af\in\mathscr{H}(X)$ for all $a\in\mathbb{C}$. However, if $f, g\in\mathscr{H}(X)$, we will see that $f+g$ need not be in $\mathscr{H}(X)$, so $\mathscr{H}(X)$ may not be a linear space. In fact, Corollary 5.3 will show that $af+bg\in\mathscr{H}(X)$ for all $a, b\in\mathbb{C}$ if and only if $f\,s\,g$. Since $af+bg$ is an amplitude superposition, s describes a superposition relation.

Example 5.1. Let $(X,\mathscr{A},\Sigma)$ be the entity of Example 3.3. Define $f,g\in\mathscr{D}(X)$ by $f(S)=g(T)=1$, and $f(T)=g(S)=0$. Then $\hat{f}(x_1)=\hat{f}(x_3)=\hat{g}(x_2)=\hat{g}(x_3)=1$ and $\hat{f}(x_2)=\hat{g}(x_1)=0$. Hence,

$$\langle\hat{f},\hat{g}\rangle_E=0\neq 1=\langle\hat{f},\hat{g}\rangle_F$$

so $f\not{s}\,g$. We now show that $h=f+g\notin\mathscr{H}(X)$. Indeed, $h(S)=h(T)=1$ and $\hat{h}(x_1)=\hat{h}(x_2)=1$, $\hat{h}(x_3)=2$. Hence,

$$\sum_{x\in E}|\hat{h}(x)|^2=2\neq 4=\sum_{x\in F}|\hat{h}(x)|^2$$

Letting $u \in \mathscr{H}(X)$ with $u \equiv 0$, we have $f\,s\,u$ and $u\,s\,g$, so s is not transitive. We now show that $\mathscr{N}(X) = \{u\}$. If $v \in \mathscr{N}(X)$, then

$$0 = \|v\|^2 = \|\hat{v}\|_E^2 = |\hat{v}(x_1)|^2 + |\hat{v}(x_2)|^2 = |\hat{v}(S)|^2 + |\hat{v}(T)|^2$$

Hence, $v(S) = v(T) = 0$, so $v = u$. ■

Example 5.2. Let $(X, \mathscr{A}, \Sigma)$ be the firefly entity of Example 3.1. Define $f \in \mathscr{H}(X)$ by $f(S) = -f(T) = -f(U) = f(V) = 1$. Then f is not identically zero, yet $\hat{f} = 0$. Hence, $f \in \mathscr{N}(X)$. We now show that $\mathscr{N}(X) = \{af : a \in \mathbb{C}\}$. Indeed, if $g \in \mathscr{N}(X)$, then in the notation of Example 3.1 we have

$$a + c = b + d = c + d = a + b = 0$$

Hence, $a = -c = -b = d$. It follows that $g = af$. ■

Lemma 5.1. Let $f, g : \Sigma \to \mathbb{C}$ and $a, b \in \mathbb{C}$. If f, g are summable, then $af + bg$ is summable and $(af + bg)^\wedge = a\hat{f} + b\hat{g}$.

Proof. Straightforward.

Theorem 5.2. Let $f, g \in \mathscr{H}(X)$. Then $f\,s\,g$ if and only if $f + g$, $f + ig \in \mathscr{H}(X)$.

Proof. Since $f, g \in \mathscr{H}(X)$, by Lemma 5.1, $f + g$ is summable and for every $E \in \mathscr{A}$ we have

$$\begin{aligned}\sum_{x \in E} |(f+g)^\wedge(x)|^2 &= \sum_{x \in E} |\hat{f}(x) + \hat{g}(x)|^2 \\ &= \|f\|^2 + \|g\|^2 + 2 \operatorname{Re} \sum_{x \in E} \hat{f}(x)\overline{\hat{g}(x)} \end{aligned} \tag{5.2}$$

By Schwarz's inequality, the summations in (5.2) are finite. Now if $f\,s\,g$, then we conclude from (5.2) that $f + g \in \mathscr{H}(X)$. Moreover, $f\,s\,(ig)$, so $f + ig \in \mathscr{H}(X)$ Conversely, if $f + g \in \mathscr{H}(X)$, then from (5.2) we have

$$\operatorname{Re}\langle \hat{f}, \hat{g} \rangle_E = \operatorname{Re}\langle \hat{f}, \hat{g} \rangle_F$$

for all $E, F \in \mathscr{A}$. If, in addition, $f + ig \in \mathscr{H}(X)$, then since

$$\begin{aligned}\sum_{x \in E} |(f+ig)^\wedge(x)|^2 &= \sum_{x \in E} |\hat{f}(x) + i\hat{g}(x)|^2 \\ &= \|f\|^2 + \|g\|^2 + 2 \operatorname{Im}\langle f, g \rangle_E\end{aligned}$$

we have

$$\operatorname{Im}\langle f, g \rangle_E = \operatorname{Im}\langle f, g \rangle_F$$

for all $E, F \in \mathscr{A}$. It follows that $f\,s\,g$. ■

Corollary 5.3. For $f, g \in \mathscr{H}(X)$, $f\,s\,g$ if and only if $af + bg \in \mathscr{H}(X)$ for all $a, b \in \mathbb{C}$.

For $f, g \in \mathscr{H}(X)$, we write $f \sim g$ if $f-g \in \mathscr{N}(X)$.

Corollary 5.4. (1) The following statements are equivalent. (a) $f \sim g$, (b) $\hat{f}=\hat{g}$, and (c) $f\, s\, g$ and $f_E = g_E$ for some $E \in \mathscr{A}$. (2) $\mathscr{N}(X)$ is a complex linear space. (3) $\sim$ is an equivalence relation.

Proof. (1) To prove (a) $\Rightarrow$ (b), suppose $f \sim g$. By Lemma 5.1, we have $\hat{f}-\hat{g}=(f-g)^{\wedge}=0$, so $\hat{f}=\hat{g}$. To prove (b) $\Rightarrow$ (c), suppose $\hat{f}=\hat{g}$. Then for every $E \in \mathscr{A}$, we have $\langle \hat{f}, \hat{g} \rangle_E = \| f \|^2$, so $f\, s\, g$. Moreover,

$$f_E = \hat{f} \mid E = \hat{g} \mid E = g_E$$

To prove (c) $\Rightarrow$ (a), suppose (c) holds. Then, by Theorem 5.2, $f-g \in \mathscr{H}(X)$ and

$$\| f-g \| = \| (f-g)_E \|_E = \| f_E - g_E \|_E = 0$$

Hence, $f \sim g$.

(2) It is clear that $0 \in \mathscr{N}(X)$ and if $f, g \in \mathscr{N}(X)$, then $af+bg \in \mathscr{N}(X)$ for all $a, b \in \mathbb{C}$.

(3) It is clear that $\sim$ is reflexive and symmetric. To prove transitivity, suppose $f \sim g$ and $g \sim h$. Then by Part (2) we have

$$f-h=(f-g)+(g-h) \in \mathscr{N}(X) \quad \blacksquare$$

For $A \subseteq \mathscr{H}(X)$ we write

$$A^s = \{ f \in \mathscr{H}(X) : f\, s\, g \text{ for all } g \in A \}$$

We call $A \subseteq \mathscr{H}(X)$ an *s-set* if $A \subseteq A^s$. Thus, A is an s-set if and only if $f\, s\, g$ for all $f, g \in A$. It is clear that singleton sets are s-sets and hence every $f \in \mathscr{H}(X)$ is contained in an s-set. Moreover, by Zorn's lemma, every s-set is contained in a maximal s-set. We denote the collection of maximal s-sets by $\mathscr{M}(X)$ and we call the elements of $\mathscr{M}(X)$ *sectors*. A sector is a maximal set of amplitudes for which superpositions are allowed. They correspond to superselection sectors for a physical system. Let $M \in \mathscr{M}(X)$. If $f \in M$ and $a \in \mathbb{C}$, then $af\, s\, g$ for all $g \in M$. Since M is maximal, $af \in M$. If $f, g \in M$, then by Theorem 5.2, $f+g \in \mathscr{H}(X)$. Also, it is clear that $(f+g)\, s\, h$ for all $h \in M$. Again by maximality, $f+g \in M$. Hence, M is a linear space. Moreover, by Corollary 5.4, $\mathscr{N}(X)$ is a subspace of M. For $f \in M$, denote the equivalence class $f+\mathscr{N}(X)$ by $[f]$ and define $\langle [f], [g] \rangle = \langle f, g \rangle$. It is straightforward to show that this is well-defined and gives an inner product on $M/\mathscr{N}(X)$. In the sequel, we shall delete the bracket on $[f]$ and simply denote $M/\mathscr{N}(X)$ by M. In this way, M becomes an inner product space. We say that $X=(X, \mathscr{A}, \Sigma)$ is *complete* if every $M \in \mathscr{M}(X)$ is a Hilbert space. We shall show that if there exists a finite $E \in \mathscr{A}$ or if $X=(X, \mathscr{A}, \Sigma(\mathscr{A}))$ is semiclassical, then X is complete. Moreover, it follows from Theorem 7 of Gudder (1986) that if $\Omega(\mathscr{A})$ is induced, then X is complete. If X is complete, $\mathscr{H}(X)$ becomes a *partial Hilbert space* (Gudder, 1986, 1988*a*). In general, for $M \in \mathscr{M}(X)$ we define dim M as the cardinality of any maximal orthonormal set in M.

We now investigate the sector structure of $\mathscr{H}(X)$. The simplest case is when X is classical and $\Sigma = \Sigma(\mathscr{A})$. In this case $\mathscr{H}(X)$ can be identified with $l^2(X)$ and $\mathscr{H}(X)$ is itself the only sector. In general, for $M \in \mathscr{M}(X)$ and $E \in \mathscr{A}$, define $U_E^M: M \to l^2(E)$ by $U_E^M f = f_E$. Then U_E^M is a linear transformation and

$$\langle U_E^M f, U_E^M g \rangle = \langle f, g \rangle$$

Hence, U_E^M is a unitary transformation from M into $\mathscr{H}_E = l^2(E)$. We thus have the following result.

Lemma 5.5. If $M \in \mathscr{M}(X)$, then dim $M \leq |E|$ for every $E \in \mathscr{A}$.

The next result improves Lemma 5.5 for the case of a semiclassical entity with a test of smallest cardinality.

Theorem 5.6. Let $X = (X, \mathscr{A}, \Sigma(\mathscr{A}))$ be semiclassical and suppose $E \in \mathscr{A}$ satisfies $|E| \leq |F|$ for every $F \in \mathscr{A}$. Then, for every $M \in \mathscr{M}(X)$ we have dim $M = |E|$ and M is complete.

Proof. Fix $M \in \mathscr{M}(X)$. By Lemma 5.5, dim $M \leq |E|$. Letting g_λ, $\lambda \in \Lambda$, be an orthonormal basis for M, we have $|\Lambda| \leq |E|$. For $F \in \mathscr{A}$, let $g_{\lambda,F} = U_F^M g_\lambda$. Then $\{g_{\lambda,F}: \lambda \in \Lambda\}$ is an orthonormal set in $\mathscr{H}_F$. Since $|E| \leq |F|$, there exists a unitary transformation $U_F: \mathscr{H}_E \to \mathscr{H}_F$ such that $U_F g_{\lambda,E} = g_{\lambda,F}$, $\lambda \in \Lambda$. Hence, $g_{\lambda,F} = U_F U_E^M g_\lambda$, $\lambda \in \Lambda$. Let $h_0 \in \mathscr{H}_E$ and define $h: X \to \mathbb{C}$ by $h|F = U_F h_0$ for every $F \in \mathscr{A}$. Now $\mathscr{H}(X) \neq \{0\}$ since there are dispersion-free amplitude densities in $\mathscr{H}(X)$, for example. Hence, dim $M \geq 1$. If $|E| = 1$, we are finished, so suppose $|E| \geq 2$. Since X is semiclassical, for every $x \in X$ there exists an $S_x \in \Sigma(\mathscr{A})$ such that $S_x \subseteq [x]$ but $S_x \nsubseteq [y]$ for every $y \in X$ with $y \neq x$. Define $f: \Sigma(\mathscr{A}) \to \mathbb{C}$ as follows. If $x \in E$, $f(S_x) = h_0(x)$; if $x \in F$, $F \neq E$, $f(S_x) = (U_F h_0)(x)$; $f(S) = 0$, otherwise. Then $\hat{f}(x) = h_0(x)$ for $x \in E$, and $\hat{f}(x) = (U_F h_0)(x)$ for $x \in F$, $F \neq E$. Hence,

$$\sum_{x \in F} |\hat{f}(x)|^2 = \|U_f h_0\|^2 = \|h_0\|^2 = \sum_{x \in E} |\hat{f}(x)|^2$$

We conclude that $f \in \mathscr{H}(X)$. Also, $f \, s \, g_\lambda$, $\lambda \in \Lambda$, since

$$\langle \hat{f}, \hat{g}_\lambda \rangle_F = \langle U_F h_0, U_F U_E^M g_\lambda \rangle_F = \langle h_0, U_E^M g_\lambda \rangle_E = \langle \hat{f}, \hat{g}_\lambda \rangle_E$$

It follows that $f \in M^s$ and since M is maximal, $f \in M$. Since $U_E^M f = h_0$, $U_E^M: M \to \mathscr{H}_E$ is surjective. Hence, dim $M = |E|$ and M is complete. ∎

The *center* of $\mathscr{H}(X)$ is defined as

$$Z[\mathscr{H}(X)] = \{f \in \mathscr{H}(X) : f\, s\, g \text{ for all } g \in \mathscr{H}(X)\} = \mathscr{H}(X)^s$$

It is straightforward to show that $Z[\mathscr{H}(X)]$ is a closed subspace of every sector.

Theorem 5.7. If $\Sigma_d \subseteq \Sigma$, then $\dim Z[\mathscr{H}(X)] \geq |\cap \mathscr{A}|$.

Proof. If $\cap \mathscr{A} = \varnothing$, we are finished. Otherwise, for $x \in \cap \mathscr{A}$, let $S_x = \{x\}$. Then clearly $S_x \in \Sigma_d \subseteq \Sigma$. Let $f_x \in \mathscr{D}(X)$ be the corresponding dispersion-free amplitude, $f_x(S_x) = 1$, $f(S) = 0$, $S \neq S_x$. Then $\hat{f}_x(x) = 1$ and $\hat{f}_x(y) = 0$, $y \neq x$. If $g \in \mathscr{H}(X)$ and $E \in \mathscr{A}$, we have $\langle \hat{g}, \hat{f}_x \rangle = \hat{g}(x)$. Hence, $f_x\, s\, g$, so $f_x \in Z[\mathscr{H}(X)]$ for all $x \in \mathscr{A}$. Moreover, the f_x, $x \in \cap \mathscr{A}$, are mutually orthogonal. Hence, $\dim Z[\mathscr{H}(X)] \geq |\cap \mathscr{A}|$. ■

For $x, y \in X$ we write $x \perp y$ if $x \neq y$ and there exists an $E \in \mathscr{A}$ such that $x, y \in E$. We say that $\mathscr{H}(X)$ is *strong* if $x \not\perp y$ implies that there exists an $f \in \mathscr{D}(X)$ such that $\hat{f}(x) = \hat{f}(y) = 1$ and if $x \neq y$, $x \not\perp y$ implies that there exists a $g \in \mathscr{D}(X)$ such that $\hat{g}(x) = -\hat{g}(y) = 1$.

Example 5.3. Let $(X, \mathscr{A}, \Sigma(\mathscr{A}))$ be semiclassical with $|E| \geq 2$ for every $E \in \mathscr{A}$. We shall show that $\mathscr{H}(X)$ is strong. If $x \neq y$ and $x \not\perp y$, then there exist $E, F \in \mathscr{A}$, $E \neq F$, such that $x \in E$ and $y \in F$. Hence, there exists an $S \in \Sigma_d$ such that $S \subseteq [x] \wedge [y]$. Define $f: \Sigma \to \mathbb{C}$ by $f(S) = 1$ and $f(T) = 0$, $T \neq S$. Then $f \in \mathscr{D}(X)$ and $\hat{f}(x) = \hat{f}(y) = 1$. For each $z \in S$ there exists $S_z \in \Sigma(\mathscr{A})$ such that $S_z \subseteq [z]$ and $S_z \nsubseteq [z']$, $z' \neq z$. Define $g: \Sigma \to \mathbb{C}$ by $g(S_x) = 1 = -g(S_y)$ and for each $G \in \mathscr{A}$ select a $z \in G$ and let $g(S_z) = 1$. Moreover, let $g(S) = 0$ otherwise. Then $g \in \mathscr{D}(X)$ and $\hat{g}(x) = -\hat{g}(y) = 1$. ■

Theorem 5.8. If $\Sigma_d \subseteq \Sigma$ and $\mathscr{H}(X)$ is strong, then

$$\dim Z[\mathscr{H}(X)] = |\cap \mathscr{A}|$$

Proof. Suppose $x \notin \cap \mathscr{A}$. We now show that there is a $y \neq x$ such that $y \not\perp x$. Assuming otherwise, we have $x \perp (X \setminus \{x\})$. Let $E \in \mathscr{A}$ with $x \notin E$. Since $\mathscr{H}(X)$ is strong, there exists a $g \in \mathscr{D}(X)$ such that $\hat{g}(x) = 1$. Since $x \perp E$, $\hat{g}(y) = 0$ for all $y \in E$. But then $\|g\| = 0$, which is a contradiction. Since $\mathscr{H}(X)$ is strong, there exist $h, h' \in \mathscr{D}(X)$ such that.

$$\hat{h}(x) = \hat{h}(y) = \hat{h}'(x) = -\hat{h}'(y) = 1$$

Now let $f \in Z[\mathscr{H}(X)]$. Since $f\, s\, h$, we have

$$\hat{f}(x)\hat{h}(x) = \hat{f}(y)\hat{h}(y)$$

so $\hat{f}(x) = \hat{f}(y)$. Moreover, $f\, s\, h'$, so

$$\hat{f}(x)\hat{h}'(x) = \hat{f}(y)\hat{h}'(y)$$

Hence, $\hat{f}(x) = -\hat{f}(y)$, so $\hat{f}(x) = 0$ for all $x \notin \cap\mathscr{A}$. If $\cap\mathscr{A} = \varnothing$, then $f = 0$. Otherwise, for $x \in \cap\mathscr{A}$, construct $f_z \in Z[\mathscr{H}(X)]$ as in the proof of Theorem 5.7. Now $\sum_{z \in \cap\mathscr{A}} |\hat{f}(z)|^2 < \infty$ and

$$\hat{f} = \sum_{z \in \cap\mathscr{A}} \hat{f}(x) f_z$$

Hence, $\{f_z : z \in \cap\mathscr{A}\}$ is an orthonormal basis for $Z[\mathscr{H}(X)]$. We conclude that $\dim Z[\mathscr{H}(X)] \leq |\cap\mathscr{A}|$ and the result follows from Theorem 5.7 ■

It is sometimes useful to embed $\mathscr{H}(X)$ in a Hilbert space and we now give a construction that does this in certain cases. We say that $\mathscr{H}(X)$ is *embeddable* if there exists a Hilbert space $\mathscr{H}_0$ and an injection $\phi: \mathscr{H}(X) \to \mathscr{H}_0$ such that $\overline{\text{sp}}\, \phi[\mathscr{H}(X)] = \mathscr{H}_0$ and $\phi | M$ is a linear unitary transformation for every $M \in \mathscr{M}(X)$.

Theorem 5.9. If $|\mathscr{A}|$ is countable then $\mathscr{H}(X)$ is embeddable.

Proof. Let $\tilde{\mathscr{S}}(\Sigma)$ be the set of summable functions on Σ. Then $\tilde{\mathscr{S}}(\Sigma)$ is a complex linear space under pointwise addition and scalar multiplication and the set of null amplitudes $\mathscr{N}(X)$ is a linear subspace of $\tilde{\mathscr{S}}(\Sigma)$. Letting $\mathscr{S}(\Sigma) = \tilde{\mathscr{S}}(\Sigma)/\mathscr{N}(X)$, we have $\mathscr{H}(X) \subseteq \mathscr{S}(\Sigma)$. Let $\mathscr{A} = \{E_1, E_2, \ldots\}$ and let $a_i > 0$ satisfy $\sum a_i = 1$. (If $|\mathscr{A}| = n < \infty$, then let $i = 1, \ldots, n$; otherwise, $i = 1, 2, \ldots$.) Let

$$\mathscr{H} = \left\{ f \in \mathscr{S}(\Sigma) : a_1 \sum_{x \in E_1} |\hat{f}(x)|^2 + a_2 \sum_{x \in E_2} |\hat{f}(x)|^2 + \cdots < \infty \right\}$$

For $f, g \in \mathscr{H}$, define

$$\langle f, g \rangle_0 = \sum a_i \sum_{x \in E_i} \hat{f}(x) \overline{\hat{g}(x)}$$

This exists and is finite, since, letting

$$b_i = \left(\sum_{x \in E_i} |\hat{f}(x)|^2 \right)^{1/2}, \qquad c_i = \left(\sum_{x \in E_i} |\hat{g}(x)|^2 \right)^{1/2}$$

we have by Schwarz's inequality

$$\begin{aligned} |\langle f, g \rangle_0| &\leq \sum a_i \sum_{x \in E_i} |\hat{f}(x)| |\hat{g}(x)| \\ &\leq \sum a_i b_i c_i \\ &= \sum (a_i^{1/2} b_i)(a_i^{1/2} c_i) \\ &\leq (\sum a_i b_i^2)^{1/2} (\sum a_i c_i^2)^{1/2} < \infty \end{aligned}$$

It is now easy to show that $(\mathscr{H}, \langle \cdot, \cdot \rangle_0)$ is an inner product space. Now $\mathscr{H}(X) \subseteq \mathscr{H}$, since, for $f \in \mathscr{H}(X)$, we have

$$\sum a_i \sum_{x \in E_i} |\hat{f}(x)|^2 = \sum a_i \|f\|^2 = \|f\|^2 < \infty$$

Let $\phi: \mathscr{H}(X) \to \mathscr{H}$ be the inclusion identity map and let $\mathscr{H}_0 = \overline{\text{sp}}\, \mathscr{H}(X) \subseteq \mathscr{H}$. To show that $\mathscr{H}(X)$ is embedded in $\mathscr{H}_0$, suppose $f, g \in \mathscr{H}(X)$ with $f\, s\, g$. Then

$$\langle f, g \rangle_0 = \sum a_i \sum_{x \in E_i} \hat{f}(x)\overline{\hat{g}(x)} = \sum a_i \langle f, g \rangle = \langle f, g \rangle \quad \blacksquare$$

It appears to be quite difficult to characterize the sector structure for an arbitrary $\mathscr{H}(X)$, although, as we shall now see, this can be done for specific examples. If $f\, s\, h$ and $\langle f, g \rangle = 0$, we write $f \perp g$.

Example 5.4. We shall characterize the sectors of the Wright triangle. Let $(X, \mathscr{A}, \Sigma)$ be the entity with $X = \{x_1, \ldots, x_6\}$, $\mathscr{A} = \{E, F, G\}$ (where $E = \{x_1, x_2, x_3\}$, $F = \{x_3, x_4, x_5\}$, and $G = \{x_1, x_5, x_6\}$, and $\Sigma = \{S_1, S_2, S_3, S_4\}$ (where $S_1 = \{x_2, x_5\}$, $S_2 = \{x_1, x_4\}$, $S_3 = \{x_3, x_6\}$, and $S_4 = \{x_2, x_4, x_6\}$). When we considered the Wright triangle in Example 2.5 we only included the states S_1, S_2, S_3. As we shall see, in that case $\mathscr{H}(X) = Z[\mathscr{H}(X)]$, so there is only one sector, namely $\mathscr{H}(X)$ itself, and dim $\mathscr{H}(X) = 3$. With the addition of S_4, the situation becomes much more interesting. It is easy to check that $\Sigma = \Sigma_d$. Let $f \in \mathscr{H}(X)$ and suppose $f(S_i) = a_i \in \mathbb{C}$, $i = 1, 2, 3, 4$. We then have $\hat{f}(x_1) = a_2$, $\hat{f}(x_2) = a_1 + a_4$, $\hat{f}(x_3) = a_3$, $\hat{f}(x_4) = a_2 + a_4$, $\hat{f}(x_5) = a_1$, and $\hat{f}(x_6) = a_3 + a_4$. Moreover, since $\|f_E\| = \|f_F\| = \|f_G\|$, we have

$$|a_2|^2 + |a_1 + a_4|^2 = |a_1|^2 + |a_2 + a_4|^2$$
$$|a_3|^2 + |a_1 + a_4|^2 = |a_1|^2 + |a_3 + a_4|^2$$
$$|a_3|^2 + |a_2 + a_4|^2 = |a_2|^2 + |a_3 + a_4|^2$$

This is equivalent to

$$\text{Re}(a_1 - a_2)\bar{a}_4 = \text{Re}(a_1 - a_3)\bar{a}_4 = \text{Re}(a_2 - a_3)\bar{a}_4 = 0 \tag{5.3}$$

If $a_4 = 0$, we call f a *type* 1 *amplitude* and if $a_4 \neq 0$, f is a *type* 2 *amplitude*. We say that $M \in \mathscr{M}(X)$ is of *type* 1 if every $f \in M$ is of type 1; otherwise, M is of *type* 2. It follows from Lemma 5.5 that dim $M \leq 3$ for every $M \in \mathscr{M}(X)$.

Lemma 5.10. A sector $M \in \mathscr{M}(X)$ is 3-dimensional if and only if M is type 1.

Proof. Since $f\, s\, g$ for any f, g of type 1, there is precisely one type 1 sector. This sector is generated by any three linearly independent type 1 amplitudes and is 3-dimensional. Conversely, suppose $M\in\mathcal{M}(X)$ is 3-dimensional and let f_1, f_2, f_3 be an orthonormal basis for M. If f_1, f_2, f_3 are all type 1, then any linear combination of them is type 1, so M is type 1. Suppose f_1, f_2 are type 1 and f_3 is type 2. Letting $f_1(S_i)=a_i$, $f_2(S_i)=b_i$, and $f_3(S_i)=c_i$, $i=1,2,3,4$, we have $a_4=b_4=0$, $c_4\neq 0$. Let $a=(a_1, a_3, a_2)$ and $b=(b_1, b_3, b_2)\in\mathbb{C}^3$, and let $d=(d_1, d_3, d_2)\in\mathbb{C}^3$ be a unit vector that is orthogonal to both a and b. Since $f_1\perp f_2$, a and b are mutually orthogonal. Since $f_3\perp f_1$ and $f_3\perp f_2$, we have (c_1, c_3+c_4, c_2) orthogonal to a and b. Hence, there exists $\alpha_3\in\mathbb{C}$, $|\alpha_3|=1$, such that

$$c_1=\alpha_3 d_1, \qquad c_3+c_4=\alpha_3 d_3, \qquad c_2=\alpha_3 d_2$$

Similarly, there exist α_2, $\alpha_1\in\mathbb{C}$, $|\alpha_2|=|\alpha_1|=1$, such that

$$c_1=\alpha_2 d_1, \qquad c_2+c_4=\alpha_2 d_2, \qquad c_3=\alpha_2 d_3$$

$$c_2=\alpha_1 d_2, \qquad c_1+c_4=\alpha_1 d_1, \qquad c_3=\alpha_1 d_3$$

If $d_3\neq 0$, then $\alpha_1=\alpha_2$. Similarly, if $d_2\neq 0$, $\alpha_1=\alpha_3$ and if $d_1\neq 0$, $\alpha_2=\alpha_3$. Hence, if $d_3\neq 0$, $c_2+c_4=\alpha_1 d_2=c_2$, which implies $c_4=0$, a contradiction. Therefore, $d_3=0$. Similarly, $d_2=d_1=0$, which is a contradiction. Hence, if f_1 and f_2 are type 1, then f_3 is type 1. Suppose f_1 is type 1 and f_2, f_3 are type 2. Again, let $f_2(S_i)=b_i$, $f_3(S_i)=c_i$, $i=1,2,3,4$. Then b_4, $c_4\neq 0$. Then letting

$$g=f_2=\frac{b_4}{c_4}f_3 \tag{5.4}$$

we have $g\in M$ and g is type 1. Also, $g\neq 0$ and $f_1\perp g$. Hence, M has an orthonormal basis with two type 1 elements. This reduces to the previous case. If f_1, f_2, f_3 are all type 2, then again defining g as in (5.4), we have $g\in M$ is of type 1. This reduces to the previous case. ∎

It follows that there is precisely one type 1 sector and this is the only 3-dimensional sector. The next lemma characterizes the type 2 sectors.

Theorem 5.11. Every type 2 sector is 2-dimensional and has the form $M=\mathrm{sp}\{f, g\}$, where f is type 1 with $f(S_i)=1$, $i=1,2,3$, while g is type 2 and satisfies $\sum_{i=1}^{4} g(S_i)=0$.

Proof. Let g be a type 2 amplitude, where $f(S_i)=b_i$, $i=1,2,3,4$. We shall show that there exists an $f\in\mathcal{H}(X)$ such that $f\perp g$. We can assume that $b_4=1$, since otherwise we could consider $b_4^{-1}g$. Let $b=\bar{b}_1^2+\bar{b}_2^2+\bar{b}_3^2$. Assume $\bar{b}_1+\bar{b}_2+\bar{b}_3\neq -1$ and let

$$c=(1-b)/(\bar{b}_1+\bar{b}_2+\bar{b}_3+1)$$

Let $f(S_i)=a_i$, $i=1, 2, 3, 4$, where $a_i=-c-\bar{b}_i$, $i=1, 2, 3$, $a_4=1$. Since $b_4=1$, it follows from (5.3) that $\text{Re}\, b_1=\text{Re}\, b_2=\text{Re}\, b_3$. Hence $\text{Re}\, a_1=\text{Re}\, a_2=\text{Re}\, a_3$; so (5.3) is satisfied, which implies that $f\in\mathscr{H}(X)$. Now

$$\begin{aligned}\langle \hat{f}, \hat{g}\rangle_E &= a_2\bar{b}_2+(a_1+a_4)(\bar{b}_1+\bar{b}_4)+a_3\bar{b}_3\\ &= a_1\bar{b}_1+a_2\bar{b}_2+a_3\bar{b}_3+a_1+\bar{b}_1+1\\ &= -c(\bar{b}_1+\bar{b}_2+\bar{b}_3)-b-c+1=0\end{aligned}$$

Similarly, $\langle \hat{f}, \hat{g}\rangle_F=\langle \hat{f}, \hat{g}\rangle_G=0$. Hence, $f\perp g$. Now suppose $\bar{b}_1+\bar{b}_2+\bar{b}_3=-1$. Let $a_i=1$, $i=1, 2, 3$, $a_4=0$. Then

$$\begin{aligned}\langle \hat{f}, \hat{g}\rangle_E &= a_2\bar{b}_2+a_1(\bar{b}_1+\bar{b}_4)+a_3\bar{b}_3\\ &= \bar{b}_1+\bar{b}_2+\bar{b}_3+1=0\end{aligned}$$

Similarly, $\langle \hat{f}, \hat{g}\rangle_F=\langle \hat{f}, \hat{g}\rangle_G=0$. Thus, in both cases we have $f\perp g$. Hence, if M is a type 2 sector, $\dim M\geq 2$. Applying Lemma 5.10, we have $\dim M=2$.

Now let M be an arbitrary type 2 sector. Then M has the form $M=\text{sp}\{f, g\}$, where $f\perp g$. We can assume that f is type 1 and g is type 2. Indeed, if f and g are type 1, then M would be type 1, which contradicts Lemma 5.10. If f and g are type 2, we can find a linear combination h of f and g that is nonzero and of type 1. Take $u\in M$ such that $u\neq 0$ and $u\perp h$. Then $M=\text{sp}\{h, u\}$. We thus have $M=\text{sp}\{f, g\}$, where f is type 1, g is type 2, and $f\perp g$. Again, letting $f(S_i)=a_i$ and $g(S_i)=b_i$, $i=1, 2, 3, 4$, we have

$$\begin{aligned}\langle \hat{f}, \hat{g}\rangle_E &= a_2\bar{b}_2+a_1(\bar{b}_1+\bar{b}_4)+a_3\bar{b}_3=0\\ \langle \hat{f}, \hat{g}\rangle_F &= a_1\bar{b}_1+a_2(\bar{b}_2+\bar{b}_4)+a_3\bar{b}_3=0\\ \langle \hat{f}, \hat{g}\rangle_G &= a_1\bar{b}_1+a_3(\bar{b}_3+\bar{b}_4)+a_2\bar{b}_2=0\end{aligned}$$

It follows that $a_1=a_2=a_3$ and $b_1+b_2+b_3+b_4=0$. Without loss of generality, we can assume that $a_1=a_2=a_3=1$. ∎

Let f be the type 1 amplitude in Lemma 5.11. It follows from Lemmas 5.10 and 5.11 that for M_1, $M_2\in\mathscr{M}(X)$ with $M_1\neq M_2$ we have $M_1\cap M_2=\text{sp}\{f\}$. Moreover, $Z[\mathscr{H}(X)]=\text{sp}\{f\}$. Notice that X is not strong, since there does not exist a $g\in\mathscr{D}(X)$ satisfying $\hat{g}(x_1)=-\hat{g}(x_4)=1$. This shows that the strongness condition cannot be deleted from Theorem 5.8.

We now consider sectors in a direct sum $X_1\oplus X_2$. Since null amplitudes are identified with the zero amplitude, it follows from the proof of Theorem 4.2 that $f\in\mathscr{H}(X_1\oplus X_2)$ if and only if $f=f_1\oplus f_2$ where $f_i\in\mathscr{H}(X_i)$, $i=1, 2$. Moreover, $\hat{f}(x)=\hat{f}_1(x)$ for every $x\in X_1$ and $\hat{f}(x)=\hat{f}_2(x)$ for every $x\in X_2$. It is clear that the components f_1, f_2 of f are unique to within a null amplitude.

Lemma 5.12. If f, $g \in \mathscr{H}(X_1 \oplus X_2)$, then $f s g$ if and only if $f_1 s g_1$ and $f_2 s g_2$.

Proof. Suppose $f_1 s g_1$ and $f_2 s g_2$. For $E_1 \in \mathscr{A}_1$, $E_2 \in \mathscr{A}_2$ and $E = E_1 \cup E_2$ we have

$$\sum_{x \in E} \hat{f}(x)\overline{\hat{g}(x)} = \sum_{x \in E_1} \hat{f}_1(x)\overline{\hat{g}_1(x)} + \sum_{x \in E_2} \hat{f}_2(x)\overline{\hat{g}_2(x)}$$

and the right-hand side is independent of E_1, E_2. Hence, $f s g$. Conversely, if $f s g$ and E_1, $E_1' \in \mathscr{A}_1$, then for a fixed $E_2 \in \mathscr{A}_2$ we have

$$\begin{aligned}\sum_{x \in E_1} \hat{f}_1(x)\overline{\hat{g}_1(x)} &= \sum_{x \in E_1 \cup E_2} \hat{f}(x)\overline{\hat{g}(x)} - \sum_{x \in E_2} \hat{f}_2(x)\overline{\hat{g}_2(x)} \\ &= \sum_{x \in E_1' \cup E_2} \hat{f}(x)\overline{\hat{g}(x)} - \sum_{x \in E_2} \hat{f}_2(x)\overline{\hat{g}_2(x)} \\ &= \sum_{x \in E_1'} \hat{f}_1(x)\overline{\hat{g}_1(x)}\end{aligned}$$

Hence, $f_1 s g_1$ and similarly $f_2 s g_2$. ∎

For $\mathscr{A}_i \subseteq \mathscr{H}(X_i)$, $i = 1, 2$, we use the notation

$$A_1 \oplus A_2 = \{f_1 \oplus f_2 : f_1 \in A_1, f_2 \in A_2\}$$

Theorem 5.13. $M \subseteq \mathscr{H}(X_1 \oplus X_2)$ is a sector if and only if $M = M_1 \oplus M_2$, where $M_i \in \mathscr{M}(X_i)$, $i = 1, 2$.

Proof. Suppose

$$M \subseteq \mathscr{H}(X_1 \oplus X_2)$$

is a sector. Let $M_i = \{f_i : f \in M\} \subseteq \mathscr{H}(X_i)$, $i = 1, 2$. Then $M \subseteq M_1 \oplus M_2$ and by Lemma 5.12, $M_i \subseteq M_i^s$, $i = 1, 2$. Suppose $g_1 \in M_1^s$. If $g_2 \in M_2$, let $g = g_1 \oplus g_2$. Then, by Lemma 5.12, $g \in M^s = M$. Hence, $g_1 \in M_1$. Therefore, $M_1 = M_1^s$, so $M_1 \in \mathscr{M}(X_1)$ and similarly $M_2 \in \mathscr{M}(X_2)$. Finally, if $g_i \in M_i$, $i = 1, 2$, then

$$g = g_1 \oplus g_2 \in M^s = M$$

Hence, $M = M_1 \oplus M_2$. Conversely, let $M_i \in \mathscr{M}(X_i)$, $i = 1, 2$, and let $M = M_1 \oplus M_2$. By Lemma 5.12, $M \subseteq M^s$. Let $g \in M^s$, $g = g_1 \oplus g_2$. By Lemma 5.12, $g_i \in M_i^s = M_i$, $i = 1, 2$. Hence, $g \in M$. Therefore, $M = M^s$, so

$$M \in \mathscr{M}(X_1 \oplus X_2) \quad ∎$$

For inner product spaces $\mathscr{H}_1$, $\mathscr{H}_2$, let $\mathscr{H}_1 \hat{\oplus} \mathscr{H}_2$ be the usual inner product space direct sum. That is,

$$\mathscr{H}_1 \hat{\oplus} \mathscr{H}_2 = \{(\psi_1, \psi_2), \psi_1 \in \mathscr{H}_1, \psi_2 \in \mathscr{H}_2\}$$

where addition and scalar multiplication are defined componentwise and

$$\langle(\psi_1, \psi_2), (\phi_1, \phi_2)\rangle = \langle\psi_1, \phi_1\rangle + \langle\psi_2, \phi_2\rangle$$

Lemma 5.14. If $M_i \in \mathcal{M}(X_i)$, $i=1, 2$, then the map

$$J: M_1 \oplus M_2 \to M_1 \hat{\oplus} M_2$$

given by $J(f_1 \oplus f_2) = (f_1, f_2)$ is an isomorphism.

Proof. Clearly, J is a bilinear bijection. Moreover, for $E = E_1 \cup E_2 \in \mathscr{A}$ we have

$$\begin{aligned}\langle f_1 \oplus f_2, g_1 \oplus g_2\rangle &= \sum_{x \in E} (f_1 \oplus f_2)^\wedge(x)\overline{(g_1 \oplus g_2)^\wedge(x)} \\ &= \sum_{x \in E_1} \hat{f}_1(x)\overline{\hat{g}_1(x)} + \sum_{x \in E_2} \hat{f}_2(x)\overline{\hat{g}_2(x)} \\ &= \langle(f_1, f_2), (g_1, g_2)\rangle = \langle J(f_1 \oplus f_2), J(g_1 \oplus g_2)\rangle \quad \blacksquare\end{aligned}$$

Finally, we consider the Cartesian product $X_1 X_2$. Recall that

$$f \in \mathscr{H}(X_1, X_2)$$

is a product amplitude if $f = f_1 f_2$, $f_i \in \mathscr{H}(X_i)$, $i = 1, 2$.

Theorem 5.15. Let $f, g \in \mathscr{H}(X_1 X_2)$ be product amplitudes. Then $f \, s \, g$ if and only if one of the following conditions holds: (a) $f_1 \, s \, g_1$ and $f_2 \, s \, g_2$, (b) $f_1 \perp g_1$, (c) $f_2 \perp g_2$.

Proof. It is clear that $f \, s \, g$ if and only if

$$\langle \hat{f}_1, \hat{g}_1\rangle_{E_1} \langle \hat{f}_2, \hat{g}_2\rangle_{E_2} = \langle \hat{f}_1, \hat{g}_1\rangle_{F_1} \langle \hat{f}_2, \hat{g}_2\rangle_{F_2} \tag{5.5}$$

for every $E_1, F_1 \in \mathscr{A}_1$, $E_2, F_2 \in \mathscr{A}_2$. If (a), (b), or (c) holds, then (5.5) holds, so $f \, s \, g$. Conversely, suppose $f \, s \, g$. If $f_1 \not\perp g_1$ and $f_2 \not\perp g_2$ then there exists an $E_1 \in \mathscr{A}_1$ such that $\langle f_1, g_1\rangle_{E_1} \neq 0$. Letting $F_1 = E_1$ and applying (5.5) gives $f_2 \, s \, g_2$. Similarly, $f_1 \, s \, g_1$. Now suppose $f_1 \not s \, g_1$. Then there exist $E_1, F_1 \in \mathscr{A}$ such that $\langle \hat{f}_1, \hat{g}_1\rangle_{E_1} \neq \langle \hat{f}_1, \hat{g}_1\rangle_{F_1}$. Letting $F_2 = E_2$ and applying (5.5) gives $\langle \hat{f}_2, \hat{g}_2\rangle_{E_2} = 0$. Hence, $f_2 \perp g_2$. Similarly, $f_2 \not s \, g_2$ implies $f_1 \perp g_1$. ■

For $M_i \in \mathcal{M}(X_i)$, $i = 1, 2$, we define

$$M_1 M_2 = \{f_1 f_2 : f_i \in M_i, i = 1, 2\}$$

It follows from Theorem 5.15 that $M_1 M_2$ is contained in a sector of $\mathscr{H}(X_1 X_2)$. In general, $M_1 M_2$ is not itself a sector. However, $M = \overline{\text{sp}}\, M_1 M_2$ is a sector and we call M a *product sector*.

6. SYSTEMS OF COVARIANCE

Let $X=(X, \mathscr{A}, \Sigma)$ be an entity and let $M\in\mathscr{M}(X)$. In the sequel, we shall assume that X is complete. This assumption is made for simplicity. It is not absolutely necessary, since the results of this section could be proved by working with the completion of M. If $E\in\mathscr{A}$, we have seen that the map $U_E^M\colon M \to \mathscr{H}_E$ defined by $U_E^M f=f_E$ is a unitary transformation (not necessarily surjective). Moreover, $A \mapsto \chi_A$ is a projection-valued (PV) measure from $\mathscr{E}(E)$ to $\mathscr{H}_E$. We then have for $A\in\mathscr{E}(E)$.

$$P_{E,f}(A)=\sum_{x\in A} |\hat{f}(x)|^2=\langle \chi_A U_E^M f, U_E^M f\rangle$$

In this sense, E is represented by the PV measure $A \mapsto \chi_A$ on $\mathscr{H}_E$.

We would now like to represent E on M. Since $U_E^M M$ is a closed subspace of $\mathscr{H}_E$, we have

$$\mathscr{H}_E=U_E^M M\hat{\oplus}(U_E^M M)^\perp$$

Define $V_E^M\colon \mathscr{H}_E \to M$ by $V_E^M=(U_E^M)^{-1}$ on $U_E^M M$ and by $V_E^M=0$ on $(U_E^M M)^\perp$. Letting P_E^M be the orthogonal projection of $\mathscr{H}_E$ onto $U_E^M M$, it is clear that $U_E^M V_E^M=P_E^M$ and $V_E^M U_E^M=I$. For $A\in\mathscr{E}(E)$, define $Q_E^M(A)\colon M \to M$ by

$$Q_E^M(A)=V_E^M\chi_A U_E^M=(U_E^M)^{-1}P_E^M\chi_A U_E^M$$

Then $Q_E^M(A)$ is a linear operator and for every $f\in M$ we have

$$\begin{aligned}\langle Q_E^M(A)f, f\rangle&=\langle V_E^M\chi_A U_E^M f, f\rangle=\langle P_E^M\chi_A U_E^M f, U_E^M f\rangle\\&=\langle \chi_A U_E^M f, U_E^M f\rangle=P_{E,f}(A)\geq 0\end{aligned}$$

It follows that $Q_E^M(A)$ is a positive operator on M. Moreover, if $A\in\mathscr{E}(F)$, we conclude that $Q_E^M(A)=Q_F^M(A)$. We also have that $A \mapsto Q_E^M(A)$ is a positive operator-valued (POV) measure from $\mathscr{E}(E)$ to M. Indeed.

$$Q_E^M(E)=V_E^M\chi_E U_E^M=V_E^M U_E^M=I$$

and if $A_i\in\mathscr{E}(E)$ are mutually disjoint, we have

$$\begin{aligned}Q_E^M(\cup A_i)&=V_E^M\chi_{\cup A_i}U_E^M=V_E^M\sum\chi_{A_i}U_E^M\\&=\sum V_E^M\chi_{A_i}U_E^M=\sum Q_E^M(A_i)\end{aligned}$$

where convergence is in the strong operator topology.

We have thus represented E by the POV measure $Q_E^M(A)$ on M. The advantage of this representation is that we can consider all tests simultaneously on the same Hilbert space M. The disadvantage is that we have replaced a PV measure by a POV measure. In the literature, POV measures are frequently called unsharp observables and PV measures are called sharp observables (Davies, 1976; Holevo, 1982; Prugovečki, 1984).

Theorem 6.1. The following statements are equivalent. (1) Q_E^M is a PV measure. (2) $P_E^M\chi_A=\chi_A P_E^M$ for all $A\in\mathscr{E}(E)$. (3) $\chi_A U_E^M=P_E^M\chi_A U_E^M$ for all $A\in\mathscr{E}(E)$

Proof. For notational simplicity, we delete the subscript E and the superscript M. To prove (2) $\Rightarrow$ (3), multiply (2) on the right by U. To prove (3) $\Rightarrow$ (1), suppose (3) holds. Then for every $A\in\mathscr{E}(E)$ we have

$$Q(A)^2=V\chi_A UV\chi_A U=V\chi_A P\chi_A U=V\chi_A U=Q(A)$$

Hence, $Q(A)$ is a projection. Moreover, $Q(A)$ is self-adjoint, since it is positive. To prove (1) $\Rightarrow$ (2), suppose Q is a PV measure. For $A\in\mathscr{E}(E)$ and $f\in M$, let $\chi_A Uf=f_1+f_2$, where $f_1\in UM$, $f_2\in(UM)^\perp$. Then $Q(A)f=Vf_1$ and

$$Q(A)^2f=V\chi_A UQ(A)f=V\chi_A UVf_1=V\chi_A f_1$$

Hence,

$$V\chi_A f_1=Q(A)f=Vf_1$$

Thus, $P(f_1-\chi_A f_1)=0$, so that

$$\chi_{A^c}f_1=f_1-\chi_A f_1\in(UM)^\perp$$

Since $f_1\in UM$, we have

$$0=\langle f_1,\chi_{A^c}f_1\rangle=\|\chi_{A^c}f_1\|^2$$

Hence, $\chi_{A^c}f_1=0$, so $\chi_A f_1=f_1$. Since $f_1=P\chi_A Uf$, we have

$$P\chi_A Uf=\chi_A P\chi_A Uf$$

Since this holds for all $f\in M$, we have

$$P\chi_A P=\chi_A P\chi_A P \tag{6.1}$$

for all $A\in\mathscr{E}(E)$. Hence,

$$P\chi_{A^c}P=\chi_{A^c}P\chi_{A^c}P \tag{6.2}$$

Adding (6.1) and (6.2) gives

$$P=2\chi_A P\chi_A P+P-\chi_A P-P\chi_A P$$

Hence, applying (6.1) gives $\chi_A P=P\chi_A P$. Taking adjoints gives $P\chi_A=P\chi_A P$, so (2) holds. ■

Let $E(M) \subseteq E$ be defined by

$$E(M) = \{x \in E: \hat{f}(x) \neq 0 \text{ for some } f \in M\}$$

and let $\hat{E}(M) = \chi_{E(M)} \mathscr{H}_E$. Then $\hat{E}(M)$ is a closed subspace of $\mathscr{H}_E$ and $U_E^M M \subseteq \hat{E}(M)$. Thus, we can consider U_E^M as a map from M into $\hat{E}(M)$.

Corollary 6.2. Q_E^M is a PV measure if and only if $U_E^M: M \to \hat{E}(M)$ is surjective.

Proof. Suppose $U_E^M M = \hat{E}(M)$. Then $P_E^M = \chi_{E(M)}$, so for every $A \in \mathscr{E}(E)$ we have

$$P_E^M \chi_A = \chi_{E(M)} \chi_A = \chi_A \chi_{E(M)} = \chi_A P_E^M$$

Applying Theorem 6.1, we find that Q_E^M is a PV measure. Conversely, suppose Q_E^M is a PV measure and let $x \in E(M)$. Then there is an $f \in M$ such that $U_E^M f(x) \neq 0$. Applying Theorem 6.1(3) gives

$$\chi_{\{x\}} = \chi_{\{x\}} U_E^M \left[\frac{f}{\hat{f}(x)} \right] = P_E^M \chi_{\{x\}} U_E^M \left[\frac{f}{\hat{f}(x)} \right] \in U_E^M M$$

Hence,

$$\hat{E}(M) = \overline{\text{sp}} \{\chi_{\{x\}}: x \in E(M)\} \subseteq U_E^M M$$

We conclude that $U_E^M M = \hat{E}(M)$. ■

Example 6.1. Let $(X, \mathscr{A}, \Sigma)$ be the entity with $X = \{x_1, x_2, y_1, y_2, y_3\}$, $\mathscr{A} = (E, F)$ (where $E = \{x_1, x_2\}$ and $F = \{y_1, y_2, y_3\}$), and

$$\Sigma = \{S_1, S_2, S_3, S_4\}$$

(where $S_1 = \{x_1, y_1\}$, $S_2 = \{x_2, y_1, y_2\}$, $S_3 = \{x_1, x_2, y_2\}$, and $S_4 = \{x_1, x_2, y_3\}$). Define $f_1, f_2: \Sigma \to \mathbb{C}$ by $f_1(S_1) = 1$, $f_1(S) = 0$, $S \neq S_1$, and $f_2(S_1) = 0$, $f_2(S_2) = 1$, and $f_2(S_3) = f_2(S_4) = 1/\sqrt{2}$. Then $\hat{f}_1(x_1) = \hat{f}_1(y_1) = 1$, $\hat{f}_1(x) = 0$, $x \neq x_1, y_1$, $\hat{f}_2(x_1) = \hat{f}_2(y_1) = 0$, $\hat{f}_2(x_2) = 1$, and $\hat{f}_2(y_2) = \hat{f}_2(y_3) = 1/\sqrt{2}$. Clearly, $f_1, f_2 \in \mathscr{D}(X)$. Since $f_1 \perp f_2$, it follows from Lemma 5.5 that $M = \text{sp}\{f_1, f_2\}$ is a sector in $\mathscr{H}(X)$. Since $U_E^M: M \to \mathscr{H}_E = \hat{E}(M)$ is surjective, it follows from Corollary 6.2 that Q_E^M is a PV measure. In fact.

$$Q_E^M(\{x_1\}) f_1 = f_1, \qquad Q_E^M(\{x_2\}) f_2 = f_2$$
$$Q_E^M(\{x_1\}) f_2 = Q_E^M(\{x_2\}) f_1 = 0$$

Denoting the 1-dimensional projection onto the subspace spanned by f as P_f, we have $Q_E^M(\{x_1\}) = P_{f_1}$ and $U_E^M(\{x_2\}) = P_{f_2}$. However, $U_F^M: M \to \mathscr{H}_F = \hat{F}(M)$ is not surjective, since

$$\dim U_F^M M = 2 < 3 = \dim \hat{F}(M)$$

Hence, by Corollary 6.2, Q_F^M is a POV measure but not a PV measure. In fact,

$$\begin{aligned}
&Q_F^M(\{y_1\})f_1 = f_1\\
&Q_F^M(\{y_1\})f_2 = V_F^M \chi_{\{y_1\}} U_F^M f_2 = V_F^M \chi_{\{y_1\}} (f_2)_F = 0\\
&Q_F^M(\{y_2\})f_1 = Q_F^M(\{y_3\})f_1 = 0\\
&Q_F^M(\{y_2\})f_2 = V_F^M \chi_{\{y_2\}} U_F^M (f_2)_F = (U_F^M)^{-1} P_F^M \chi_{\{y_2\}} (f_2)_F\\
&\qquad = (U_F^M)^{-1} \tfrac{1}{2} (f_2)_F = \tfrac{1}{2} f_2\\
&Q_F^M(\{y_3\})f_2 = \tfrac{1}{2} f_2
\end{aligned}$$

Hence,

$$Q_F^M(\{y_1\}) = P_{f_1},\ Q_F^M(\{y_2\}) = (1/2)P_{f_2}$$

and

$$Q_F^M(\{y_3\}) = (1/2)P_{f_2} \quad \blacksquare$$

In order to consider interference in the present context, we must extend the definition of Q_F^M. We say that $A \in \mathscr{E}(\mathscr{A})$ is (E, f)-bounded if $\hat{f}(A) | E \in \mathscr{H}_E$. If A is (E, f)-bounded, we define $Q_E^M(A)f = V_E^M[\hat{f}(A) | E]$. This reduces to the usual definition of Q_E^M, since for $A \subseteq E$ we have $\hat{f}(A) | E = \chi_A f_E$, so

$$Q_E^M(A)f = V_E^M \chi_A f_E = V_E^M \chi_A U_E^M f$$

Theorem 6.3. Let $E, F \in \mathscr{A}$ and $f \in M$ and suppose U_E^M is surjective. Then E does not interfere with F relative to f if and only if every $A \in \mathscr{E}(F)$ is (E, f)-bounded and

$$\| Q_E^M(A) f \|^2 = \langle Q_F^M(A) f, f \rangle \tag{6.3}$$

Proof. Suppose E does not interfere with F relative to f. Then for $A \in \mathscr{E}(F)$ we have

$$\| \hat{f}(A) \|_E^2 = P_{E,f}(A) = P_{F,f}(A) < \infty$$

Hence, A is (E,f)-bounded. Moreover, since U_E^M is surjective,

$$\begin{aligned}\|Q_E^M(A)f\|^2 &= \|(U_E^M)^{-1}\hat{f}(A)\|_E^2 = \|\hat{f}(A)\|_E^2 \\ &= P_{F,f}(A) = \langle Q_F^M(A)f, f\rangle\end{aligned}$$

Conversely, if every $A\in\mathscr{E}(F)$ is (E,f)-bounded and (6.3) holds, we have

$$P_{E,f}(A) = \|Q_E^M(A)f\|^2 = \langle Q_F^M(A)f, f\rangle = P_{F,f}(A)$$

Hence, E does not interfere with F relative to f. ■

If $A\in\mathscr{E}(\mathscr{A})$ is (E,f)-bounded for every $f\in M$, then $Q_E^M(A)$ becomes a linear operator on M. We can then obtain the following stronger result.

Corollary 6.4. If U_E^M is surjective, then E does not interfere with F relative to every $f\in M$ if and only if every $A\in\mathscr{E}(F)$ is (E,f)-bounded for all $f\in M$ and we have

$$Q_E^M(A)^* Q_E^M(A) = Q_F^M(A)$$

Proof. By Theorem 6.3, E does not interfere with F relative to every $f\in M$ if and only if (6.3) holds for all $f\in M$, $A\in\mathscr{E}(F)$. But then

$$\|Q_E^M(A)f\|^2 \leq \|Q_F^M(A)\|\,\|f\|^2 \leq \|f\|^2$$

so $Q_E^M(A)$ is a bounded linear operator on M for all $A\in\mathscr{E}(F)$. Moreover, for all $f\in M$ and $A\in\mathscr{E}(F)$, we have

$$\langle Q_E^M(A)^* Q_E^M(A)f, f\rangle = \|Q_E^M(A)f\|^2 = \langle Q_F^M(A)f, f\rangle$$

The result now follows. ■

If g_1 and g_2 are maps from X into X, we denote their composition by $g_1 \circ g_2$. A *symmetry group* on an entity $(X, \mathscr{A}, \Sigma)$ is a group G of bijections $g: X \to X$ with group operation $g_1 g_2 = g_1 \circ g_2$ such that for every $g\in G$, $E\in\mathscr{A}$, and $S\in\Sigma$ we have

$$gE = \{gx : x\in E\} \in \mathscr{A}$$

$$gS = \{gx : x\in S\} \in \Sigma$$

Notice that $g: \mathscr{A} \to \mathscr{A}$ and $g: \Sigma \to \Sigma$ are bijections. If a group of bijections G satisfies $gE\in\mathscr{A}$ for every $E\in\mathscr{A}$ and $g\in G$, then G automatically preserves supports, so the condition $gS\in\Sigma$ for every $S\in\Sigma$ and $g\in G$ is not as strong as it first appears. Indeed, suppose S is a support, $g\in G$, E, $F\in\mathscr{A}$. Since a bijection preserves inclusions and intersections, if $(gS)\cap E\subseteq F$, then $S\cap(g^{-1}E)\subseteq g^{-1}F$. Since S is a support, $S\cap(g^{-1}F)\subseteq g^{-1}E$. Hence, $(gS)\cap F\subseteq E$, so gS is a support. In particular, if G is a group of bijections on X such that $gE\in\mathscr{A}$ for all $g\in G$ and $E\in\mathscr{A}$, then G is a symmetry group on $(X, \mathscr{A}, \Sigma(\mathscr{A}))$. In the sequel, G will denote a symmetry group on an entity $(X, \mathscr{A}, \Sigma)$.

Lemma 6.5. For $x \in X$, $S \in \Sigma$, and $g \in G$, $S \subseteq [x]$ if and only if $gS \subseteq [gx]$.

Proof. Suppose $S \subseteq [x]$. Then there exists an $E \in \mathscr{A}$ such that $S \cap E = x$. Since a bijection preserves intersections, $gS \cap gE = gx$. Hence, $gS \subseteq [gx]$. Applying g^{-1} to this result gives the converse. ■

For $f \in \mathscr{H}(X)$ and $g \in G$, define $U_g f: \Sigma \to \mathbb{C}$ by $U_g f(S) = f(g^{-1}S)$.

Lemma 6.6. (1) The map U_g is a bijection on $\mathscr{H}(X)$ satisfying $U_{g_1 g_2} = U_{g_1} U_{g_2}$ for all $g_1, g_2 \in G$. (2) If $M \in \mathscr{M}(X)$, then $U_g M \in \mathscr{M}(X)$ and U_g is a unitary transformation from M onto $U_g M$.

Proof. (1) For $f \in \mathscr{H}(X)$, we have by Lemma 6.5 that

$$(U_g f)^\wedge(x) = \sum_{S \subseteq [x]} U_g f(S) = \sum_{S \subseteq [x]} f(g^{-1}S) = \sum_{g^{-1}S \subseteq [g^{-1}x]} f(g^{-1}S)$$
$$= \hat{f}(g^{-1}x) \tag{6.4}$$

For $E \in \mathscr{A}$, we have

$$\sum_{x \in E} |(U_g f)^\wedge(x)|^2 = \sum_{x \in E} |\hat{f}(g^{-1}x)|^2 = \sum_{g^{-1}x \in g^{-1}E} |\hat{f}(g^{-1}x)|^2 = \|f\|^2$$

Hence, $U_g f \in \mathscr{H}(X)$. It is clear that U_g is injective. To show that U_g is surjective, suppose $h \in \mathscr{H}(X)$. Define $f: \Sigma \to \mathbb{C}$ by $f(S) = h(gS)$. Then $f \in \mathscr{H}(X)$ and $U_g f = h$. Finally, for $g_1, g_2 \in G$, we have

$$U_{g_1 g_2} f(S) = f(g_2^{-1} g_1^{-1} S) = U_{g_2} f(g_1^{-1} S) = U_{g_1} U_{g_2} f(S)$$

(2) Suppose $f_1 \, s \, f_2$. Then for any $E \in \mathscr{A}$, applying (6.4), we have

$$\sum_{x \in E} (U_g f_1)^\wedge(x)\overline{(U_g f_2)^\wedge(x)} = \sum_{x \in E} \hat{f}_1(g^{-1}x)\overline{\hat{f}_2(g^{-1}x)}$$
$$= \sum_{g^{-1}x \in g^{-1}E} \hat{f}_1(g^{-1}x)\overline{\hat{f}_2(g^{-1}x)}$$
$$= \langle \hat{f}_1, \hat{f}_2 \rangle_{g^{-1}E} = \langle f_1, f_2 \rangle$$

Hence, $U_g f_1 \, s \, U_g f_2$. It easily follows that $U_g M \in \mathscr{M}(X)$. The above also shows that U_g is a unitary transformation from M onto $U_g M$. ■

For $M \in \mathscr{M}(X)$, we write $gM = U_g M \in \mathscr{M}(X)$. We call $g \mapsto U_g$ a *generalized unitary representation of* G. Let $E \in \mathscr{A}$, $h \in \mathscr{H}_E$. For $x \in gE \in \mathscr{A}$, define $\bar{U}_g h(x) = h(g^{-1}x)$. It follows from (6.4) that $(U_g f)^\wedge(x) = \bar{U}_g \hat{f}(x)$ for all $x \in E$. As in the proof of Lemma 6.6, $\hat{U}_g$ is a unitary operator from $\mathscr{H}_E$ onto $\mathscr{H}_E$ satisfying $\hat{U}_{g_1 g_2} = \hat{U}_{g_1} \hat{U}_{g_2}$ for all $g_1, g_2 \in G$. Hence, $g \mapsto \hat{U}_g$ is a unitary representation of G. We say that $g\in G$ *leaves* $M\in\mathcal{M}(X)$ *invariant* if $gM=M$. We use the notation $\hat{M}=\{\hat{f}:f\in M\}$.

Corollary 6.7. $gM=M$ if and only if $\hat{U}_g\hat{M}\subseteq\hat{M}$.

Proof. Suppose $gM=M$ and let $\hat{f}\in\hat{M}$. Then $U_gf\in M$, so

$$\hat{U}_g\hat{f}=(U_gf)^\wedge\in\hat{M}$$

Hence, $\hat{U}_g\hat{M}\subseteq\hat{M}$. Conversely, suppose $\hat{U}_g\hat{M}\subseteq\hat{M}$ and let $f\in M$. Then $\hat{f}\in\hat{M}$, so $(U_gf)^\wedge=\hat{U}_g\hat{f}\in\hat{M}$. Hence, $U_gf\in M$ and $gM\subseteq M$. Since $gM\in\mathcal{M}(X)$, $gM=M$. ■

The next result shows that Q_E^M is a generalized system of covariance for U_g.

Theorem 6.8. For every $g\in G$, $M\in\mathcal{M}(X)$, $E\in\mathcal{A}$, and $A\in\mathcal{E}(E)$ we have

$$U_g^{-1}Q_E^{gM}(A)U_g=Q_{g^{-1}E}^{M}(g^{-1}A) \tag{6.5}$$

Proof. We first show that

$$U_E^{gM}U_g=\hat{U}_gU_{g^{-1}E}^{M} \tag{6.6}$$

Letting $f\in M$ and $x\in E$, we have

$$(U_E^{gM}U_gf)(x)=(U_gf)^\wedge(x)=\hat{U}_g\hat{f}(x)=(\hat{U}_gU_{g^{-1}E}^{M}f)(x)$$

so (6.6) holds. We now show that

$$\hat{U}_gP_{g^{-1}E}^{M}=P_E^{gM}\hat{U}_g \tag{6.7}$$

Let $h\in U_{g^{-1}E}^{M}M$. Then $h=U_{g^{-1}E}^{M}f$, $f\in M$, and by (6.6),

$$\hat{U}_gh=\hat{U}_gU_{g^{-1}E}^{M}f=U_E^{gM}U_gf\in U_E^{gM}gM$$

Thus,

$$P_E^{gM}\hat{U}_gh=\hat{U}_gh=\hat{U}_gP_{g^{-1}E}^{M}h$$

Now suppose $h\in(U_{g^{-1}E}^{M}M)^\perp$. Then $\hat{U}_gP_{g^{-1}E}^{M}h=0$. Let $h'\in U_E^{gM}gM$. Then $h'=U_E^{gM}f'$, $f'\in gM$. Hence, by (6.6),

$$\hat{U}_g^{-1}h'=\hat{U}_{g^{-1}}U_E^{gM}f'=U_{g^{-1}E}^{M}U_{g^{-1}}f'\in U_{g^{-1}}^{M}M$$

Hence,

$$\langle\hat{U}_gh,h'\rangle=\langle h,\hat{U}_g^{-1}h'\rangle=0$$

so that $\hat{U}_gh\in(U_E^{gM}gM)^\perp$. Hence, $P_E^{gM}\hat{U}_gh=0$ and (6.7) holds.

We next show that

$$U_g V_{g^{-1}E}^{M} = V_E^{gM} \hat{U}_g \tag{6.8}$$

By (6.6) we have

$$U_g = V_E^{gM} \hat{U}_g U_{g^{-1}E}^{M}$$

Hence, applying (6.7) gives

$$U_g V_{g^{-1}E}^{M} = V_E^{gM} \hat{U}_g P_{g^{-1}E}^{M} = V_E^{gM} P_E^{gM} \hat{U}_g = V_E^{gM} \hat{U}_g$$

We now show that

$$\chi_A U_E^{gM} U_g = \hat{U}_g \chi_{g^{-1}A} U_{g^{-1}E}^{M} \tag{6.9}$$

Letting $f \in \mathscr{H}(X)$, we have, by (6.6),

$$\begin{aligned}(\chi_A U_E^{gM} U_g f)(x) &= \chi_A(x)(U_E^{gM} U_g f)(x)\\ &= \chi_A(x)(\hat{U}_g U_{g^{-1}E}^{M} f)(x)\\ &= \chi_{g^{-1}A}(g^{-1}x)(U_{g^{-1}E}^{M} f)(g^{-1}x)\\ &= (\hat{U}_g \chi_{g^{-1}A} U_{g^{-1}E}^{M} f)(x)\end{aligned}$$

Finally, applying (6.9) and (6.8) gives

$$\begin{aligned}Q_E^{gM}(A) U_g &= V_E^{gM} \chi_A U_E^{gM} U_g = V_E^{gM} \hat{U}_g \chi_{g^{-1}A} U_{g^{-1}E}^{M}\\ &= U_g V_{g^{-1}E}^{M} \chi_{g^{-1}A} U_{g^{-1}E}^{M} = U_g Q_{g^{-1}E}^{M}(g^{-1}A)\end{aligned}$$

The result now follows. ■

Corollary 6.9. (1) If G leaves all sectors invariant, then

$$U_g^{-1} Q_E^{M}(A) U_g = Q_{g^{-1}E}^{M}(g^{-1}A)$$

for all $g \in G$, $M \in \mathscr{M}(X)$, $E \in \mathscr{A}$, and $A \in \mathscr{E}(E)$.

(2) If G leaves all sectors and tests invariant, then

$$U_g^{-1} Q_E^{M}(A) U_g = Q_E^{M}(g^{-1}A) \tag{6.10}$$

for all $g \in G$, $M \in \mathscr{M}(X)$, $E \in \mathscr{A}$, and $A \in \mathscr{E}(E)$.

Equation (6.10) is usually called a system of covariance for U_g. It is easy to show that G leaves all sectors invariant if and only if $f\, s\, h$ implies $f\, s\, U_g h$ for all $g \in G$. This is equivalent to the following condition. If f, $h \in M$ for any $M \in \mathscr{M}(X)$, then, for every E, $F \in \mathscr{A}$ and $g \in G$, we have

$$\sum_{x \in E} \hat{f}(x) \overline{\hat{h}(g^{-1}x)} = \sum_{x \in F} \hat{f}(x) \overline{\hat{h}(g^{-1}x)}$$

Example 6.2. Let $(X, \mathscr{A}, \Sigma)$ be the Wright triangle of Example 5.4. If we draw X as an equilateral triangle, then it is easy to see that the symmetric group on three elements $G=\{I, R, R^2, F_1, F_2, F_3\}$ is a symmetry group on X. Define $f_i \in \mathscr{D}(X)$, $i=1, \ldots, 5$, as follows: $f_i(S_j)=\delta_{ij}$, $i, j=1, 2, 3, 4$, and

$$f_5(S_1)=f_5(S_2)=f_5(S_3)=-f_5(S_4)=\frac{1}{\sqrt{2}}$$

We then have

$$\hat{f}_1(x_2)=\hat{f}_1(x_5)=\hat{f}_2(x_1)=\hat{f}_2(x_4)=\hat{f}_3(x_3)=\hat{f}_3(x_6)$$
$$=\hat{f}_4(x_2)=\hat{f}_4(x_4)=\hat{f}_4(x_6)=1$$
$$\hat{f}_5(x_1)=\hat{f}_5(x_3)=\hat{f}_5(x_5)=\frac{1}{\sqrt{2}}$$

and $\hat{f}_i(x)=0$ otherwise, $i=1, \ldots, 5$. We have seen in Example 5.4 that $M_1=\text{sp}\{f_1, f_2, f_3\}$ and $M_2=\text{sp}\{f_4, f_5\}$ are sectors in $\mathscr{H}(X)$. Notice that E, F, G do not interfere with each other relative to f_i, $i=1, 2, 3, 4$, since these amplitude densities are dispersion-free. For f_5 we have

$$\hat{f}_5(\{x_3, x_4\})(x_1)=-\hat{f}_5(\{x_3, x_4\})(x_2)=\hat{f}_5(\{x_3, x_4)\}(x_3)=\frac{1}{\sqrt{2}}$$

Hence,

$$P_{E,f_5}(\{x_3, x_4\})=\tfrac{3}{2}\neq\tfrac{1}{2}=P_{F,f_5}(\{x_3, x_4\})$$

and E interferes with F relative to f_5.

Since $U_E^{M_1}$ is surjective, by Corollary 6.2, $Q_E^{M_1}$ is a PV measure. In fact, it is easy to show that $Q_E^{M_1}(\{x_1\})=P_{f_2}$, $Q_E^{M_1}(\{x_2\})=P_{f_1}$, and $Q_E^{M_1}(\{x_3\})=P_{f_3}$. In a similar way $Q_F^{M_1}$ and $Q_G^{M_1}$ are PV measures. However,

$$U_E^{M_2}M_2=\text{sp}\{\hat{f}_4|E, \hat{f}_5|E\}\neq\hat{E}_{M_2}$$

so by Corollary 6.2, $Q_E^{M_2}$ is a POV measure which is not a PV measure. In fact, it can be shown that $Q_E^{M_2}(\{x_2\})=P_{f_4}$ and

$$Q_E^{M_2}(\{x_1\})=Q_E^{M_2}(\{x_3\})=\tfrac{1}{2}P_{f_5}$$

Similar results hold for $Q_F^{M_2}$ and $Q_G^{M_2}$.

Let U_g be the generalized unitary representation of G on $\mathscr{H}(X)$ defined previously. Since G leaves M_1 invariant, we have

$$U_g^{-1}Q_H^{M_1}(A)U_g=Q_{g^{-1}H}^{M_1}(g^{-1}A)$$

for every $g \in G$, $H \in \mathscr{A}$, $A \in \mathscr{E}(H)$. We now consider M_2. Since $U_g = I$ on M_2, we have

$$Q_H^{M_2}(A) = Q_{g^{-1}H}^{M_2}(g^{-1}A)$$

for all $g \in G$, $H \in \mathscr{A}$, and $A \in \mathscr{E}(H)$.

We now show that a generalization of the Nagy extension theorem holds for this example. Since $Q_H^{M_1}$, $H \in \mathscr{A}$, are already PV measures, we need not consider them now, so we study $Q_H^{M_2}$, $H \in \mathscr{A}$. Let $\mathscr{H} = \mathbb{C}^\Sigma$ with inner product

$$\langle f, g \rangle = \tfrac{1}{3}[\langle \hat{f}, \hat{g} \rangle_E + \langle \hat{f}, \hat{g} \rangle_F + \langle \hat{f}, \hat{g} \rangle_G]$$

Then $\mathscr{H}$ is a Hilbert space and if $f\, s\, g$, this reduces to the usual inner product on $\mathscr{H}(X)$. Hence, $\mathscr{H}(X)$ is embedded in $\mathscr{H}$ and M_2 (and M_1) are closed subspaces of $\mathscr{H}$. Let P be the projection of $\mathscr{H}$ onto M_2. Notice that dim $\mathscr{H} =$ 4, since f_1, f_2, f_3, f_4 are linear independent. It is easy to show that there exists an orthonormal basis g_1, g_2, g_3, g_4 for $\mathscr{H}$ such that $g_4 = f_4$, $g_3 \perp f_5$, and

$$\langle g_1, f_5 \rangle = \langle g_2, f_5 \rangle = \frac{1}{\sqrt{2}}$$

Define the PV measure P_E from $\mathscr{E}(E)$ to $\mathscr{H}$ by

$$P_E(\{x_1\}) = P_{g_1}, \qquad P_E(\{x_2\}) = P_{g_3} + P_{g_4}, \qquad P_E(\{x_4\}) = P_{g_2}$$

Then

$$PP_E(\{x_1\}) = (P_{f_4} + P_{f_5})P_{g_1} = P_{f_5}P_{g_1}$$
$$PP_E(\{x_3\}) = (P_{f_4} + P_{f_5})P_{g_2} = P_{f_5}P_{g_2}$$
$$PP_E(\{x_2\}) = (P_{f_4} + P_{f_5})(P_{g_4} + P_{g_3}) = P_{f_4}$$

Hence,

$$PP_E(\{x_3\})f_4 = P_{f_5}P_{g_1}g_4 = 0$$

$$PP_E(\{x_1\})f_5 = P_{f_5}P_{g_1}f_5 = \frac{1}{\sqrt{2}}P_{f_5}g_1 = \frac{1}{2}f_5$$

Similarly, $PP_E(\{x_3\})f_4 = 0$ and $PP_E(\{x_3\})f_5 = \frac{1}{2}f_5$. Hence,

$$PP_E(\{x_i\})P = Q_E^{M_2}(\{x_i\}), \qquad i = 1, 2, 3$$

We conclude that $Q_E^{M_2}$ is the projection of a PV measure. A similar result holds for $Q_F^{M_2}$ and $Q_G^{M_2}$. ■

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